TRANSVERSE NONLINEAR INSTABILITY OF SOLITARY WAVES FOR SOME HAMILTONIAN PDE'S

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Abstract. We present a general result of transverse nonlinear instability of 1-d solitary waves for Hamiltonian PDE's for both periodic or localized transverse perturbations. Our main structural assumption is that the linear part of the $1d$ model and the transverse perturbation "have the same sign". Our result applies to the generalized KP-I equation, the Nonlinear Schrödinger equation, the generalized Boussinesq system and the Zakharov-Kuznetsov equation and we hope that it may be useful in other contexts.

Contents

1. Introduction

A lot of two-dimensional dispersive equations possess one-dimensional solitary waves which are stable when submitted to one-dimensional perturbations but which are destabilized when submitted to general two-dimensional perturbations. This phenomenon has been known for a long time in the physics literature. For example, by using the Lax pair structure of the KP-I equation, it was proven in [32] that the KdV solitary wave seen as a 1d solution of the KP-I equation is unstable. For non-integrable equations, the general instability theory of solitary waves of [10] does not seem to apply since the 1-d solitary wave is not a constrained critical point of the Hamiltonian of the 2d equation. Nevertheless, in some cases, the linear instability can be proven by some simple bifurcation arguments, for example, the linear instability of the 1d solitary wave of the 2d Nonlinear Schrödinger equation (NLS) can be proven by the Zakharov-Rubenchik bifurcation argument for small transverse frequencies. Consequently, it seems interesting to reduce the proof of nonlinear instability to the search for unstable eigenmode for the linearized equation by proving that linear instability implies nonlinear instability for a large class of equations.

In [27], we have shown that the method developed by Grenier [12] for the incompressible Euler equation can be adapted to prove transverse instability of solitary waves in dispersive models. More precisely, we have proven two nonlinear instability results for solitary waves of the Korteweg- de Vries and the $1d$ Nonlinear Schrödinger equations (NLS), seen as solutions of the KP-I or the $2d$ NLS equations respectively and subject to periodic transverse perturbations. The linear instability in both cases was known. More precisely, in the KP-I case one has a complete understanding of the possible unstable modes for any fixed transverse frequency while in the NLS case unstable modes where detected thanks

to the Zakharov-Rubenchik bifurcation argument for small transverse frequencies. The possibility of describing all unstable modes in the KP-I case seems to be related to the Lax pairs structure of the KP-I equation (sometimes called complete integrability). The Zakharov-Rubenchik bifurcation argument is a more general feature but does not seem to apply in some important cases such as the gKP-I equation, a case which is in the scope of the applicability of the present paper. Our goal here is to present a general transverse nonlinear instability theory of solitary waves, assuming the spectral instability of the solitary wave, for Hamiltonian PDE's obeying to some structural assumptions described below, the main one being that, in some sense, the transverse perturbation and the $1d$ dispersion operator should have the same sign. More precisely, we state two instability results, one for transverse periodic boundary condition and one where the transverse direction is unbounded and the perturbations are localized. This last case was not studied in our previous work [27] and requires more work in the study of low frequencies. We also present a criterion to detect unstable modes, and thus to prove linear instability, inspired by the work of Groves-Haragus-Sun [13], which is different and more flexible than the one presented in our previous work [27] for NLS. Finally, we check that our general theory can be applied to prove the linear and nonlinear instability of 1d solitary waves in the generalized KP-I equation, the 2d NLS equation, a Boussinesq type equation, the Zakharov-Kuznetsov equation and the KP-BBM equation.

Our method mainly depends on the Hamiltonian structure of the equation and we hope that the ideas of this paper may be extended to more general, not necessarily linear transverse perturbations. In particular, we hope that our approach may be useful to get transverse instability for some more complicated fluid mechanics models.

The paper is organized as follows. We first describe the general framework and our assumptions. Then we state two abstract instability results under the additional assumption of the existence of an unstable mode of the linearized equation. Some of our assumptions will be easily verified in the applications. Other assumptions such as the existence of multipliers or the bounded frequencies resolvent estimates are not a general feature in the considered framework. For that reason in the later sections we present criteria insuring the validity of these assumptions and in particular, a criterion for the existence of unstable eigenmodes. These criteria will be usefull to analyze our concrete examples. In the last section of the paper, we apply the general theory to various examples.

Acknowledgement. We are indebted to Jean-Claude Saut for several discussions on the subject and in particular for providing us the reference [13].

2. General framework and results

2.1. **The unperturbed model.** For s a real number, we consider the Sobolev spaces $H^s \equiv H^s(\mathbb{R};\mathbb{R}^d)$, where $d \geq 1$ is an integer and we denote its norm by $|\cdot|_s$. The L^2 norm

will be simply denoted by $|\cdot|$ and the L^2 scalar product by $(\cdot,\cdot)$. We consider the equation

$$\partial_t u = J(L_0 u + \nabla F(u)), \tag{2.1}$$

where $F \in C^\infty(\mathbb{R}^d;\mathbb{R})$, $F(0)=0$ and the linear operators J and L are such that :

- J is a Fourier multiplier which is skew-symmetric for the L^2 scalar product with domain containing H^1 (thus J is of order at most one) and such that $\mathrm{Ker}J = \{0\}$.
- L_0 is a Fourier multiplier which is a symmetric operator with a self adjoint realisation on $L^2(\mathbb{R};\mathbb{R}^d)$ with domain $D(L_0)$ containing H^2. Moreover, L_0 is coercive,

$$C^{-1}|u|_1^2 \leq (L_0 u, u) \leq C|u|_1^2\,. \tag{2.2}$$

Note that since J and L_0 are Fourier multipliers, they commute with the x derivative and hence we have that $J \in \mathcal{B}(H^s, H^{s-1})$ and $L_0 \in \mathcal{B}(H^s, H^{s-2})$ for every s.

Equation (2.1) can thus be written in the Hamiltonian form

$$\partial_t u = J\nabla H(u), \quad H(u) = \frac{1}{2}(L_0 u, u) + \int_{-\infty}^{\infty} F(u)\,dx\,.$$

One may imagine situations when J and L_0 are of higher orders. In these cases some modifications of the considered framework should be done. However, in all our examples L_0 is of order 2 and J of order 0 or 1.

We are interested in the stability of stationary solutions of (2.1). Since J is into, they are critical points of the Hamiltonian H, i.e. we have $\nabla H(Q) = L_0 Q + \nabla F(Q) = 0$. We focus on the case where Q is smooth, $Q \in H^\infty$. Next, we consider the linear operator associated to the second variation of the Hamiltonian at Q :

$$L \equiv D_u(\nabla H)(Q) = L_0 + R, \quad Ru = \nabla^2 F(Q)u.$$

Note that R is a bounded operator on H^s for every $s \geq 0$ since Q and F are smooth. Consequently, L is a self adjoint operator on L^2 with domain $D(L_0)$. Our main assumption on L is that its spectrum is under the form

$$\sigma(L) = \{\mu\} \cup \{0\} \cup \Sigma, \tag{2.3}$$

where $\mu < 0$ is a simple eigenvalue, 0 is an eigenvalue of finite multiplicity and $\Sigma \subset [\alpha, +\infty)$ for some $\alpha > 0$. Moreover, the eigenspaces associated to μ and zero are made of smooth eigenvectors (i.e which are in H^∞). Many of our arguments remain valid if $\sigma(L) \cap]-\infty, 0]$ contains a finite number of eigenvalues of finite multiplicities. We will be interested in situations where Q is a stable object for (2.1). Note that the spectral assumption (2.3) is one of the main assumption which allows to prove the stability of Q by the Grillakis-Shatah-Strauss method [10].

2.2. **The transversally perturbed model.** We are interested in the stability of Q when (2.1) can be embedded in a larger Hamiltonian equation

$$\partial_t u = \mathcal{J}(\partial_y)(L_0 u + \nabla F(u) + \mathcal{S}(\partial_y)u), \tag{2.4}$$

where now u also depends on y with $y \in \mathbb{T}_a = \mathbb{R}/2\pi a\mathbb{Z}$ or $y \in \mathbb{R}$ and L_0 acts in a natural way on functions of 2 variables. The operators $\mathcal{J}(\partial_y)$, $\mathcal{S}(\partial_y)$ are operator valued Fourier multipliers in y, i.e. if $\mathcal{F}_y$ stands for the Fourier transform in y, we have

$$\mathcal{F}_y(\mathcal{S}(\partial_y)u)(k) = S(ik)\mathcal{F}_y(u)(k), \quad \mathcal{F}_y(\mathcal{J}(\partial_y)u)(k) = J(ik)\mathcal{F}_y(u)(k).$$

Moreover, $S(ik)$ and $J(ik)$ are now Fourier multipliers in x. In the following, we still denote by $(\cdot,\cdot)$ and $|\cdot|_s$ the complex scalar product of $L^2(\mathbb{R},\mathbb{C}^d)$ and the H^s norm for complex valued functions respectively.

2.2.1. *Assumptions on the operator $J(ik)$.* For every k, $J(ik)$ is a Fourier multiplier such that:

- $J(ik)$ and $J(ik)L_0$ are skew symmetric on $L^2(\mathbb{R})$, $J(0) = J$,
- the domain of $J(ik)$ contains H^1, Ker $J(ik) = \{0\}$ and we have the uniform bound

$$\exists\, C > 0,\, \forall\, k, \quad |J(ik)u| \leq C|u|_1,\, \forall u \in H^1, \tag{2.5}$$

- The commutator $[R, J(ik)]$ is a uniformly bounded operator on L^2 :

$$\exists\, C > 0, \quad \forall\, k, \quad \big|\big([R, J(ik)]w, w\big)\big| \leq C|w|^2. \tag{2.6}$$

Note that since $J(0) = J$, $\mathcal{J}(\partial_y)u = Ju$ if u depends only on x. We also point out that the assumption (2.6) is obviously verified when J is a bounded operator on L^2.

2.2.2. *Assumptions on the operator $S(ik)$.* For every k, $S(ik)$ is a Fourier multiplier such that:

- $S(ik)$ is non-negative and symmetric, $J(ik)S(ik)$ is skew symmetric, $S(0) = 0$,
- $S(ik)$ has a self-adjoint realisation on L^2 with domain $\mathcal{D}_S$ independent of k for $k \neq 0$,
- $J(ik)S(ik)J(ik)$ and $J(ik)S(ik)\partial_x$ belong to $\mathcal{B}(H^2(\mathbb{R}), L^2(\mathbb{R}))$,
- Let us set $|w|^2_{S(ik)} \equiv (w, S(ik)w)$, then there exists a non-negative continuous function (possibly unbounded) $C(k)$ such that

$$|J(ik)S(ik)u|_{L^2} \leq C(k)|u|_{S(ik)}, \quad \forall\, u \in \mathcal{D}_S\,. \tag{2.7}$$

Note that (2.4) also has an Hamiltonian structure with Hamiltonian given by

$$\mathcal{H}(u) = \int \Big(\frac{1}{2}(L_0 u, u) + \frac{1}{2}(\mathcal{S}(\partial_y)u, u) + \int_{\mathbb{R}} F(u)\, dx\Big)\, dy.$$

Moreover, we also point out that Q is still a stationary solution of (2.4) and more generally that if u is a (reasonable) solution of (2.4) which does not depend on y, then u actually solves (2.1).

2.2.3. *Compatibility between $S(ik)$ and L.* We assume that there exists K and $c_0 > 0$ such that for every $|k| \geq K$,

$$(Lv, v) + (S(ik)v, v) \geq c_0 |v|_1^2, \quad \forall\, v \in H^2 \cap \mathcal{D}_S. \tag{2.8}$$

This is one of our main structural assumption which roughly says that S and L_0 have the same sign. This assumption is valid for example for the KP-I equation and the 2d NLS equation but not for the KP-II equation or the hyperbolic Schrödinger equation.

2.3. **The resolvent equation.** In this subsection, we state our assumptions on the linearization of (2.4) about Q. Since $\mathcal{S}(\partial_y)$ is a linear map, the linearization of (2.4) about Q reads

$$v_t = \mathcal{J}(\partial_y)(L + \mathcal{S}(\partial_y))v. \tag{2.9}$$

Definition 2.1. *An unstable mode for (2.4) is a function $U \in L^2 \cap \mathcal{D}(S(ik))$ such that for some $\sigma \in \mathbb{C}$ with $Re(\sigma) > 0$ and some $k \in \mathbb{R}$, the problem (2.9) has a solution of the form*

$$v(t, x, y) = e^{\sigma t} U(x) e^{iky}\,. \tag{2.10}$$

We call σ the amplification parameter and k the transverse frequency associated to U.

Thus if U is an unstable mode then it is a solution of the eigenvalue problem

$$\sigma U = J(ik)(L + S(ik))U, \quad U \in L^2(\mathbb{R}; \mathbb{C}^d). \tag{2.11}$$

2.3.1. *Assumption of existence of an Evans function and 1d stability.* We assume that there exists a function $D(\sigma, k)$ (Evans function) such that for every k, $D(\cdot, k)$ is analytic in $\mathrm{Re}\,\sigma > 0$ and such that there exists an unstable mode (2.10) if and only if $D(\sigma, k) = 0$. We also assume that all the possible unstable eigenmodes are smooth (H^∞) and that Q is spectrally stable with respect to one-dimensional perturbations which reads:

$$D(\sigma, 0) \neq 0, \quad \mathrm{Re}\,\sigma > 0. \tag{2.12}$$

A concrete criterion for the existence of the Evans function will be given in section 4. In most examples we have in mind, (2.11) can be reduced to an ordinary differential equation and hence, as usual, the Evans function will be defined as a Wronskian determinant associated to an ODE obtained after some manipulations from (2.11).

Next, let us consider the resolvent equation for $Re(\sigma) > 0$

$$\sigma U = J(ik)LU + J(ik)S(ik)U + J(ik)F, \quad U \in H^\infty(\mathbb{R};\mathbb{C}^d) \cap \mathcal{D}(S(ik), \ F \in H^\infty(\mathbb{R},\mathbb{C}^d)\,. \tag{2.13}$$

2.3.2. *Bounded frequencies resolvent bounds in the periodic case.* We assume that there exists q such that for every k and every $\mathcal{K}$ compact set in Re $\sigma > 0$, every $s \geq 0$, there exists $C_{k,\mathcal{K},s}$ such that if $D(\cdot, k)$ does not vanish on $\mathcal{K}$, then for every $F \in H^\infty(\mathbb{R})$, there is a unique solution $U \in H^\infty(\mathbb{R}) \cap \mathcal{D}(S(ik))$ of (2.13) which satisfies

$$|u|_s \leq C_{k,\mathcal{K},s}|F|_{s+q}, \quad \forall \sigma \in \mathcal{K}. \tag{2.14}$$

2.3.3. *Bounded frequencies resolvent bounds in the localized case.* When k is a continuous variable, we need some uniform dependence in k in the regime $k \sim 0$. We shall assume that the Evans function D is analytic in (σ, k) for Re $\sigma > 0$ and $k \neq 0$ and that there exists an analytic continuation $\tilde{D}(\sigma, k)$ which is analytic in $\{\text{Re } \sigma > 0\} \times \mathbb{R}$. Moreover, we assume a strong $1D$ stability

$$\tilde{D}(\sigma, 0) \neq 0, \quad \forall \sigma, \text{ Re } \sigma > 0 \tag{2.15}$$

and the uniform (also with respect to k) resolvent bound : there exists $q \geq 0$ such that for every compact set $\mathcal{K}$ in $\{\text{Re } \sigma > 0\}$ and $M > 0$, there exists $C_{\mathcal{K},M,s}$ such that if $\tilde{D}(\sigma, k)$ does not vanish on $\mathcal{K} \times (0, M]$, then for every $F \in H^\infty$, there is a unique solution $U \in H^\infty \cap \mathcal{D}(S(ik))$ of (2.13) which satisfies

$$|u|_s \leq C_{\mathcal{K},M,s}|F|_{s+q}, \quad \forall (\sigma, k) \in \mathcal{K} \times (0, M]. \tag{2.16}$$

As we shall see below, in most examples the existence of the Evans function and the bounds (2.14), (2.16) can be obtained by ODE techniques. We shall give below a simple criterion which allows to obtain (2.14), (2.16). We also point out that we allow the case where $\tilde{D}(\sigma, 0)$ is different from $D(\sigma, 0)$ since we have not assumed continuity of D at $k = 0$. Typically $D(\sigma, k)$ is the determinant of a matrix of fixed size for $k \neq 0$ and $D(\sigma, 0)$ is the determinant of a smaller matrix.

As we shall prove, the assumptions (2.3), (2.8) and the structural properties of the operators given in sections 2.2.1, 2.2.2 are sufficient to ensure nice resolvent bounds in the energy norm H^1 for large $|\text{Im } \sigma|$. The following assumption will be used to get the estimates of higher order derivatives.

2.3.4. *Existence of a multiplier.* We suppose that for every $s \geq 2$, there exists a self-adjoint operator M_s such that there exists $C > 0$ with

$$|(M_s u, v)| \leq C|u|_s\,|v|_s, \quad (M_s u, u) \geq |u|_s^2 - C|u|_{s-1}^2 \tag{2.17}$$

and

$$\mathrm{Re}(J(ik)(L + S(ik))u, M_s u) \leq C_k |u|_s\,|u|_{s-1}. \tag{2.18}$$

The assumption (2.18) will play a key role for the control on higher derivatives in a resolvent analysis below. In the cases of "semi-linear" problems we will be able simply to choose $M_s = \partial_x^{s-1} L \partial_x^{s-1}$.

2.4. **The nonlinear problem.** Finally, we make a set of assumptions on the nonlinear problem (2.4). Denote by $\mathbb{H}^s$ the Sobolev type spaces on $\mathbb{R}\times\mathbb{R}$ or $\mathbb{R}\times\mathbb{T}_a$ with the norms $\|\cdot\|_s$. We denote by $\|\cdot\|$ the norm of L^2. Consider the problem

$$\partial_t u = \mathcal{J}(\partial_y)(L_0 u + \nabla F(u^a + u) - \nabla F(u^a) + \mathcal{S}(\partial_y)u) + \mathcal{J}(\partial_y)G,\ \ u(0) = 0, \tag{2.19}$$

where u^a is a smooth function bounded with all its derivatives and $G \in C(\mathbb{R}; \mathbb{H}^s)$ for every s. We suppose that the problem (2.19) is locally well-posed in the sense that for every u^a and G satisfying the previous assumptions there exists a time $T > 0$ and a solution of (2.19) in $C([0,T]; \mathbb{H}^s)$ for every $s \geq s_0$ ($s_0 > 0$ being sufficiently large), unique in a suitable class. Finally, we assume the that tame estimate

$$\Big| \Big(\!\Big(\partial_x^\alpha \partial_y^\beta \mathcal{J}(\partial_y)\big(D\nabla F(w+v)\cdot v\big), \partial_x^\alpha \partial_y^\beta v \Big)\!\Big) \Big| \leq \omega\big(\|w\|_{W^{s+1,\infty}} + \|v\|_s\big)\|v\|_s^2 \tag{2.20}$$

holds for every α, β, $\alpha + \beta = s$, where ω is a continuous non-decreasing function with $\omega(0) = 0$ and $(\!(\cdot,\cdot)\!)$ is the L^2 scalar product for functions of two variables.

This last assumption together with the properties of the operators $\mathcal{J}$ and L_0 will ensure the existence of an $\mathbb{H}^s$ energy estimate for (2.19).

2.5. **Statement of the abstract results.** Let us state our first instability result for (2.4) with $\mathbb{R}\times\mathbb{T}_a$ as a spatial domain.

Theorem 1 (Nonlinear transverse periodic instability)**.** *Consider the Hamiltonian equation (2.4) and suppose that the assumptions of the previous sections hold true, except the assumptions of Sections 2.3.3. Assume also that there exists an unstable mode with corresponding transverse frequency $k_0 \neq 0$. Then we have nonlinear instability of* (2.4) *defined on $\mathbb{R}\times\mathbb{T}_{2\pi/k_0}$. More precisely for every $s \geq 0$, there exists $\eta > 0$ such that for every $\delta > 0$, there exists $u_0^\delta \in \mathbb{H}^\infty(\mathbb{R}\times\mathbb{T}_{2\pi/k_0})$ and a time $T^\delta \sim |\log\delta|$ such that $\|u_0^\delta - Q\|_s \leq \delta$ and the solution u^δ of* (2.4) *with data u_0^δ remains in $\mathbb{H}^s$ on $[0, T^\delta]$ and satisfies $d(u^\delta(T^\delta), \mathcal{F}) \geq \eta$ where $\mathcal{F}$ is the space of $L^2(\mathbb{R})$ functions depending only on x and $d(u, \mathcal{F}) = \inf_{v\in\mathcal{F}} \|u - v\|$.*

Notice that we have a strong instability statement since we measure the initial perturbation in a strong norm such as $\|\cdot\|_s$ while the instability occurs in the weaker norm L^2. Our second result concerns fully localized perturbations.

Theorem 2 (Nonlinear transverse localized instability)**.** *Consider the Hamiltonian equation (2.4) and suppose that the assumptions of the previous sections hold true. Assume also that there exists an unstable mode with $k \neq 0$. Then we have nonlinear instability of* (2.4) *posed on $\mathbb{R}^2$. More precisely for every $s \geq 0$, there exists $\eta > 0$ such that for every $\delta > 0$, there exists u_0^δ and a time $T^\delta \sim |\log \delta|$ such that $\|u_0^\delta - Q\|_s \leq \delta$ and the solution u^δ of* (2.4) *with data u_0^δ remains defined on $[0, T^\delta]$, i.e. $u^\delta - Q \in \mathbb{H}^s$, $\forall t \in [0, T^\delta]$ and satisfies $d(u^\delta(T^\delta), \mathcal{F}) \geq \eta$ where again $\mathcal{F}$ is the space of $L^2(\mathbb{R})$ functions depending only on x and $d(u, \mathcal{F}) = \inf_{v \in \mathcal{F}} \|u - v\|$.*

These theorems state that the existence of an unstable eigenmode implies nonlinear orbital instability of the solitary wave. Indeed, the orbit of Q under the action of all the possible groups of invariance of (2.1) remain in $\mathcal{F}$. In particular our results exclude the possibility of orbital stability of Q with respect to the spatial translations. More precisely our result implies that

$$\inf_{a \in \mathbb{R}} \|u(T^\delta) - Q(\cdot - a)\| \geq \eta.$$

There are many assumptions in these theorems, nevertheless, some of them will be very easy to check on examples, for example the structural assumption 2.2.1, 2.2.2. The ones which are more difficult to check are the assumptions 2.3.1, 2.3.2, 2.3.3, 2.3.4 that is to say, the existence of an Evans function and of multipliers, the bounded frequencies resolvent bounds, and also the assumption on the existence of an unstable eigenmode. Consequently, the next sections are devoted to the proof of more concrete criteria which ensure that these assumptions are verified and which are easy to test on examples.

Let us explain the main steps in the proof of Theorem 2. The inspiration comes from the work of Grenier [12] in fluid mechanics problems. We believe that this scheme is quite general and may be useful in other contexts.

1. The first step is to prove that the possible unstable modes in the sense of Definition 2.1 above necessarily belong to a compact set both with respect to the transverse frequency and the amplification parameter. This allows to find the most unstable mode i.e. with the largest real part of the amplification parameter (note that there exists at least an unstable eigenmode by assumption) and to define a first approximate growing solution by a wave packet construction in the framework of Theorem 2.
2. The second step is to evaluate, both from above and below, in a suitable norm (here it is L^2) the first approximate growing solution given by step 1. In the proof of Theorem 2, we need to use the Laplace method and some properties of the curve $k \to \sigma(k)$.

3. The third step is, following Grenier [12], the construction of a refined approximate solutions which is carefully estimated from above. Since we deal with Hamiltonian PDE's this step requires a different argument compared to similar estimates for diffusive problems. Here we reduce the matters to resolvent bounds for $\sigma - J(ik)(L+S(ik))$ for σ's with real parts larger that the amplification parameter of the most unstable mode and any k in the (compact) set of possible transverse frequencies.
4. The last step is to estimate the difference between the refined approximate solution and the true solution on the interval $[0, T^{\delta}]$ by energy estimates. The analysis in this step is quite flexible and seems to apply each time we have H^s energy estimates for the full $2d$ problem.

The paper is organized as follows. In section 3 we give a criterion for the existence of an unstable eigenmode, in section 4, we give criteria for the existence of the Evans function and the bounded frequencies resolvent bounds and in section 5, we give a criterion for the existence of multipliers satisfying (2.17), (2.18). The two next sections are devoted to the proof of Theorem 1 and 2 and finally, the last section is devoted to the study of various examples for which we check that the general theory can be applied.

3. A sufficient condition for the existence of an unstable mode

In this section, we give a simple criterion which ensures the existence of an unstable eigenmode. This criterion is inspired by the work [13]. Consider the symmetric operator defined by

$$M_k = J(ik)LJ(ik) + J(ik)S(ik)J(ik).$$

Since $J(ik)S(ik)J(ik) \in \mathcal{B}(H^2, L^2)$ by assumption 2.2.1, we get that the domain $\mathcal{D}$ of M_k contains H^4 (indeed, $J(ik)$ is at most a first order operator and L is a second order operator).

A simple criterion for the existence of an unstable eigenmode is given by the following statement.

Lemma 3.1. *Assume that for every k and every u real-valued $J(ik)u$ and $S(ik)u$ are also real valued. Next, assume that there exists $k_0 \neq 0$ such that zero is a simple eigenvalue of M_{k_0} with corresponding real-valued nontrivial eigenvalue $\varphi \in H^\infty$ normalized so that $\|\varphi\|_{L^2(\mathbb{R})} = 1$. Finally, assume that M_{k_0} is a Fredholm map of index zero, that M_k depends smoothly on k for k close to k_0 and the non degeneracy condition*

$$\Big([\frac{d}{dk} M_k]_{k=k_0}(\varphi), \varphi \Big) \neq 0\,. \tag{3.1}$$

Then there exists k in a vicinity of k_0 and $\sigma > 0$ such that there exists an unstable mode with amplification parameter σ and transverse frequency k.

As we shall see, this criterion can be used on many examples.

Proof of Lemma 3.1. We need to solve the problem

$$\sigma v = J(ik)Lv + J(ik)S(ik)v, \quad v \in L^2(\mathbb{R})$$

for k close to k_0 and σ close to 0. We shall seek for σ real and v real-valued. This is legitimate since by assumption $J(ik)v$ and $S(ik)v$ are real-valued if v is real-valued. We shall look for $k = k(\sigma)$ with $k(0) = k_0$. Since Ker $J(ik) = \{0\}$, we look for v under the form $v = J(ik)u$, $u \in L^2$. Therefore, we need to solve the problem $F(u, k, \sigma) = 0$, where

$$F(u, k, \sigma) = M_k(u) - \sigma J(ik)u\,.$$

We search for u of the form $u = \varphi + w$ with $w \in \tilde{\mathcal{D}} \equiv \{u \in \mathcal{D} \cap L^2(\mathbb{R}, \mathbb{R}^d) : (u, \varphi) = 0\}$. Define

$$G(w, k, \sigma) \equiv F(\varphi + w, k, \sigma) = M_k\varphi - \sigma J(ik)\varphi + M_k w - \sigma J(ik)w$$

as a map on $\tilde{\mathcal{D}} \times \mathbb{R} \times \mathbb{R}$ to L^2. Note that we have

$$G(0, k_0, 0) = M_{k_0}\varphi = 0$$

since φ is an eigenvector of M_{k_0} by assumption. Next for $(w, \mu) \in \tilde{\mathcal{D}} \times \mathbb{R}$, we have

$$D_{w,k}G(0, k_0, 0)[w, \mu] = M_{k_0}w + \mu\big([\frac{d}{dk}M_k]_{k=k_0}\,\varphi\big)\,.$$

Thanks to (3.1) the linear map $D_{w,k}G(0, k_0, 0)$ is a bijection from $\tilde{\mathcal{D}} \times \mathbb{R}$ to $L^2(\mathbb{R}, \mathbb{R}^d)$. Consequently, by the implicit function theorem, for σ close to 0 there exist $w(\sigma) \in \tilde{\mathcal{D}}$ and $k(\sigma) \in \mathbb{R}$ with $w(0) = 0$ and $k(0) = k_0$ such that $G(w(\sigma), k(\sigma), \sigma) = 0$. This completes the proof of Lemma 3.1. □

4. Criteria for the existence of the Evans function and the bounded frequencies resolvent bounds

In this section we describe some concrete criteria in order to ensure the assumptions of sections 2.3.1, 2.3.2 and 2.3.3. The first assumption roughly says that we can reduce the eigenvalue problem (2.11) to an ordinary differential equation.

4.1. **Reduction to an ODE.** We thus assume that there exists a Fourier multiplier $R(\sigma, k)$ such that $R(\sigma, k) \in \mathcal{B}(H^{s+l_k}, H^s))$ for every $s \geq 0$ and that Ker $R = \{0\}$. Moreover, we assume the block structure

$$\sigma R(\sigma, k) - R(\sigma, k)J(ik)\big(L + S(ik)\big) = \begin{pmatrix} P_1(\sigma, k) & 0 \\ P_2(\sigma, k) & E(\sigma, k) \end{pmatrix} \tag{4.1}$$

where :

- For every k, $P_1(\sigma,k)$ is a $r\times r$ matrix of differential operators of order $m_k\geq 1$ with coefficients which depend analytically on σ,

$$P_1(\sigma,k)=\partial_x^{m_k}\mathrm{Id}+\cdots, \tag{4.2}$$

- for every k, $P_2(\sigma,k)$ is an operator of order $\leq m_k-1$ i.e. $P_2(\sigma,k)\in\mathcal{B}(H^{s+m_k-1},H^s)$ for every $s\geq 0$
- For every k, $E(\sigma,k)$ is invertible and $E(\sigma,k)^{-1}\in\mathcal{B}(H^s,H^s)$ for every $s\geq 0$.
- There exists (l,m) such that for every $k\neq 0$, $(l_k,m_k)=(l,m)$ and $l_0\leq l$, $m_0\leq m$.

Moreover, all the operators depend continuously on σ for $\mathrm{Re}\,\sigma>0$ for each fixed k.

Because of the triangular block structure (4.1), the study of the resolvent equation (2.13) can be reduced to the study of the ordinary differential equation

$$P_1(\sigma,k)u_1=(R(\sigma,k)J(ik)F)_1 \tag{4.3}$$

by using the block decomposition $U=(u_1,u_2)^t\in\mathbb{C}^r\times\mathbb{C}^{d-r}$. Note that we allow the possibility that $r=d$, which means that the resolvent equation can be directly reduced to an ordinary differential equation by applying the operator $R(\sigma,k)$.

We can rewrite (4.3) as a first order ordinary differential equation

$$\frac{dV}{dx}=A(x,\sigma,k)V+\mathbb{F}, \tag{4.4}$$

where $A(x,\sigma,k)\in\mathcal{M}_{N_k}(\mathbb{C})$, $N_k=m_k\,r$ is a matrix which depends smoothly on x, analytically on σ and

$$\mathbb{F}=(0,\cdots,0,(R(\sigma,k)J(ik)F)_1). \tag{4.5}$$

Note that $A(x,\sigma,k)$ is in general not "continuous" at $k=0$, since for $k=0$, the dimension of the matrix may be different.

With our reduction assumptions, we have unstable eigenmodes if and only if the ODE (4.4) with $\mathbb{F}=0$ has a nontrivial L^2 solution.

4.2. **Asymptotic behavior and consistent splitting.** We add the assumption that there exist $A_\infty(\sigma,k)$ and $C>0$, $\alpha>0$, such that for every $x,k\in\mathbb{R}$, and every σ,

$$|A(x,\sigma,k)-A_\infty(\sigma,k)|\leq Ce^{-\alpha|x|}, \tag{4.6}$$

and that the spectrum of $A_\infty(\sigma,k)$ **does not** meet the imaginary axis for $Re(\sigma)>0$.

4.3. Existence of the Evans function.

Lemma 4.1. *Under the assumptions of sections 4.1, 4.2, there exists a function $D(\sigma,k)$ (Evans function) which is analytic in $Re(\sigma)>0$, for every k and such that $D(\sigma,k)=0$ if and only if there exists a non trivial eigenmode solution of* (2.11).

Proof. By classical arguments (see e.g. [1]), the assumptions of section 4.2 allows to define an Evans function $D(\sigma, k)$ for (4.4) which is an analytic function in $Re(\sigma) > 0$, for every k and such that $D(\sigma, k) = 0$ if and only if there exists a non trivial $L^2(\mathbb{R}; \mathbb{R}^{m_k r})$ solution of $V' = A(x, \sigma, k)V$ which is actually exponentially decreasing. Thanks to the reduction assumptions 4.1 above, this is equivalent to the existence of a nontrivial solution of (2.11).

4.4. **Resovent estimates in the periodic case.** Under the above assumptions, we can prove :

Lemma 4.2. *Let $R(\sigma, k)$ satisfying assumptions* (4.1) *and* (4.2)*, then, there exists $q \geq 0$ such that for every k, every $s \geq 0$ and every compact $\mathcal{K} \subset \{Re\ \sigma > 0\}$, there exists $C_{k,\mathcal{K},s}$ such that if $D(\cdot, k)$ does not vanish on $\mathcal{K}$, then there is a unique solution $U \in H^\infty \cap \mathcal{D}_S$ of* (2.13) *for every $F \in H^\infty$ which satisfies*

$$|u|_s \leq C_{k,\mathcal{K},s} |F|_{s+q}. \tag{4.7}$$

In other words, if one can prove the existence of $R(\sigma, k)$ then one get the resolvent bounds (2.14) on every compact which does not contain unstable eigenmode.

4.5. **Resolvent estimates in the localized case.** To get (2.14) in the localized case, we need some assumptions on the dependence of the various objects with respect to k. We assume that:

i) $R(\sigma, k)$, $P_2(\sigma, k)$, $E(\sigma, k)$ depend continuously on (σ, k) for $k \neq 0$, $\mathrm{Re}\ \sigma > 0$ and have continuous extensions up to $\{\mathrm{Re}\ \sigma > 0\} \times \mathbb{R}$.

ii) P_1 and thus A and A_∞ are analytic for $k \neq 0$, $\mathrm{Re}\ \sigma > 0$ and have analytic extensions up to $\{\mathrm{Re}\ \sigma > 0\} \times \mathbb{R}$.

Next, since the spectrum of $A_\infty(\sigma, k)$ does not meet the imaginary axis for $\mathrm{Re}(\sigma) > 0$ and $k \neq 0$, we can define a projection on the stable subspace of A_∞ which is analytic in σ and k by the Dunford integral

$$P_\infty(\sigma, k) = \int_\Gamma (z - A_\infty(\sigma, k))^{-1}\, dz,$$

where Γ is a contour which encloses all the negative real part eigenvalues of A_∞, the projection on the unstable subspace is then given by $Id - P_\infty$. Note that we had assumed that $A_\infty(\sigma, 0+) = \lim_{k \to 0} A_\infty(\sigma, k)$ exist but we allow the presence of eigenvalues on the imaginary axis. We nevertheless assume:

iii) the projection $P_\infty(\sigma, k)$ can be continued analytically to $\{Re(\sigma) > 0\} \times \mathbb{R}$.

This implies thanks to the Gap Lemma ([11]), [17]) that the Evans function can also be continued analytically to $\{Re(\sigma) > 0\} \times \mathbb{R}$. The continuation of the function will be denoted by $\tilde{D}(\sigma, k)$. Recall that $\tilde{D}(\sigma, 0)$ may be different from $D(\sigma, 0)$. Indeed $A(x, \sigma, k)$ is not continuous at zero and hence $A(x, \sigma, 0) \neq \lim_{k \to 0} A(x, \sigma, k)$. By construction, the same difference holds for the Evans function.

Finally, we also assume :

iv) for every compact set K of $\{Re\,\sigma > 0\}$, and every $s \geq 0$, there exists $C > 0$ such that for every eigenvalue $\mu(\sigma, k)$ of $A_\infty(\sigma, k)$

$$\begin{aligned} \|R(\sigma,k) - R(\sigma,0^+)\|_{\mathcal{B}(H^{s+l},H^s)} + \|J(ik) - J(0)\|_{\mathcal{B}(H^{s+1},H^s)} \\ + \|R(\sigma,k)J(ik)S(ik)\|_{\mathcal{B}(H^{s+m},H^s)} \leq C\rho(k,K) \end{aligned} \tag{4.8}$$

where

$$\rho(k,K) = \inf_{\sigma\in K, \mu(\sigma,k)\in SpA_\infty(\sigma,k)} |\mathrm{Re}\,\mu(\sigma,k)|$$

for every k in a small disk $D(0,r)\backslash\{0\}$ and $\sigma \in K$, where l, m and k are defined in section 4.1.

Note that since $S(0) = 0$, this assumption is nontrivial only when there exists an eigenvalue of $A_\infty(\sigma, k$ such that $\mathrm{Re}\,\mu(\sigma,k)$ vanishes at $k = 0$.

Then, we can prove the following statement.

Lemma 4.3. *Assuming the existence of $R(\sigma,k)$ given by assumptions 4.1, 4.2 and assumptions i)-iv) above, then there exists $q \geq 0$ such that for every $s \geq 0$, every compact $\mathcal{K} \subset \{Re\ \sigma > 0\}$ and $M > 0$, there exists $C_{\mathcal{K},M,s}$ such that if $\tilde{D}$ does not vanish on $\mathcal{K} \times [0,M]$ and D does not vanish on $\mathcal{K}$, then, for every $F \in H^\infty$, there is a unique solution $U \in H^\infty \cap \mathcal{D}(S(ik))$ of* (2.13) *which satisfies*

$$|u|_s \leq C_{\mathcal{K},M,s}|F|_{s+q}, \quad \forall\sigma \in \mathcal{K},\ \forall k \in (0,M]. \tag{4.9}$$

Consequently, we have given criteria which allow to obtain (2.16)

4.6. **Proof of Lemma 4.2.** By using $R(\sigma,k)$ and setting $w = (u_1,u_2)^t$, we can rewrite

$$\sigma w = J(ik)(Lw + S(ij)w) + J(ik)F$$

as

$$V_x = A(\sigma,k,x)V + \mathbb{H}, \tag{4.10}$$

and

$$u_2 = -E(\sigma,k)^{-1}P_2(\sigma,k)u_1 + E(\sigma,k)^{-1}\big(R(\sigma,k)J(ik)F\big)_2$$

with $V(x) = (u_1,\cdots,\partial_x^{m_k-1}u_1(x))$ and $\mathbb{H} = (0,\cdots,(R(\sigma,k)J(ik)F)_1)$.

The properties of E, $J(ik)$ and P_1 and the triangular structure already give

$$|u_2|_s \leq C_s\big(|u_1|_{s+m_k-1} + |F|_{l_k+s+1}\big).$$

Consequently, it suffices to prove that for every $s \geq 0$,

$$|V|_s \leq C_s|\mathbb{H}|_s$$

where V is the solution of the ODE (4.10) to get the result.

Let us denote by $T(\sigma,k,x,x')$ the fundamental solution of $V_x = \mathbb{A}V$ i.e. the solution such that $T(\sigma,k,x',x') = Id$. Thanks to our assumption (4.6) on the behavior as $|x| \to \infty$ of $A(\sigma,k,x)$, we can use classical perturbative ODE arguments (more precisely the roughness of exponential dichotomy, see [7] for example). Namely, the equation $V_x = AV$ has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$, i.e., there exists projections $P^+(\sigma,k,x)$, $P^-(\sigma,k,x)$ which are smooth in the parameter σ with the invariance property

$$T(\sigma,k,x,x')P^{\pm}(\sigma,k,x') = P^{\pm}(\sigma,k,x)T(\sigma,k,x,x') \tag{4.11}$$

and such that there exists C and $\alpha > 0$ such that for every $U \in \mathbb{C}^{N_k}$, and $\sigma \in \mathcal{K}$, we have

$$\begin{aligned}
&|T(\sigma,k,x,x')P^+(\sigma,k,x')U| \leq Ce^{-\alpha(x-x')}\ |P^+(\sigma,k,x')U|,\ x \geq x' \geq 0,\\
&|T(\sigma,k,x,x')(I-P^+(\sigma,k,x'))U| \leq Ce^{\alpha(x-x')}\ |(I-P^+(\sigma,k,x'))U|,\ 0 \leq x \leq x',\\
&|T(\sigma,k,x,x')P^-(\sigma,k,x,x')U| \leq Ce^{\alpha(x-x')}\ |P^-(\sigma,k,x')U|,\ x \leq x' \leq 0,\\
&|T(\sigma,k,x,x')(I-P^-(\sigma,k,x'))U| \leq Ce^{-\alpha(x-x')}\ |(I-P^-(\sigma,k,x'))U|,\ 0 \geq x \geq x'.
\end{aligned}$$

In particular, note that a solution $T(\sigma,k,x,0)V^0$ is decaying when x tend to $\pm\infty$ if and only if V^0 belongs to $\mathcal{R}(P^{\pm}(\sigma,k,0))$. Since when σ is in $\mathcal{K}$, the Evans function does not vanish, we have by definition no non trivial solution decaying in both sides and hence we have

$$\mathcal{R}(P^+(\sigma,k,0)) \cap \mathcal{R}(P^-(\sigma,k,0)) = \{0\}. \tag{4.12}$$

Let us choose bases $(r_1^{\pm},\cdots,r_{N^{\pm}}^{\pm})$ of $\mathcal{R}(P^{\pm}(\sigma,k,0))$ (where $N^+ + N^- = N_k$) which depends on σ in a smooth way (see [18] for example) then we can define

$$M(\sigma,k) = (r_1^+,\cdots,r_{N^+}^+,r_1^-,\cdots,r_{N^-}^-)$$

and we note that $M(\sigma,k)$ is invertible for $\sigma \in \mathcal{K}$ because of (4.12). With, these new notations, we note in passing that the Evans function can actually be defined by

$$D(\sigma,k) = \det M(\sigma,k).$$

This allows us to define a new projection $P(\sigma,k)$ by

$$P(\sigma,k) = M(\sigma,k)\begin{pmatrix} I_{N^+} & 0 \\ 0 & 0 \end{pmatrix} M(\sigma,k)^{-1}$$

and next

$$P(\sigma,k,x) = T(\sigma,k,x,0)P(\sigma,k).$$

The main interest of these definitions is that we have $\mathcal{R}(P(\sigma,k)) = \mathcal{R}(P^+(\sigma,k,0))$ and $\mathcal{R}(I-P(\sigma,k)) = \mathcal{R}(P^-(\sigma,k,0))$. Therefore thanks to (4.11), we have for every x that $\mathcal{R}(P(\sigma,k,x)) = \mathcal{R}(P^+(\sigma,k,x))$ and similarly that

$$\mathcal{R}(I-P(\sigma,k,x)) = \mathcal{R}(P^-(\sigma,k,x)).$$

Consequently, we have the estimates

$$|T(\sigma,k,x,x')P(\sigma,k,x')| \leq Ce^{-\alpha(x-x')}\ ,\ x,\ x'\in\mathbb{R},\ x\geq x',\ \forall\sigma\in\mathcal{K}, \tag{4.13}$$

$$|T(\sigma,k,x,x')(I-P(\sigma,k,x'))| \leq Ce^{\alpha(x-x')}\ ,\ x,\ x'\in\mathbb{R},\ x\leq x',\ \forall\sigma\in\mathcal{K}. \tag{4.14}$$

By using this property, the unique bounded solution of (4.10) reads by Duhamel formula

$$V(x)=\int_{-\infty}^{x} T(\sigma,k,x,x')P(\sigma,k,x')\mathbb{H}(x')\,dx' - \int_{x}^{+\infty} T(\sigma,k,x,x')(I-P(\sigma,k,x'))\mathbb{H}(x')\,dx'$$

and hence, we get thanks to (4.13), (4.14) that

$$|V(x)| \leq C\int_{\mathbb{R}} e^{-\alpha|x-x'|}|\mathbb{H}(x')|\,dx'$$

which yields by standard convolution estimates

$$|V|_{L^2} \leq C|\mathbb{H}|_{L^2},$$

We next estimate higher order derivatives. Write

$$\partial_x^{s+1}V = \mathbb{A}\partial_x^s V + [\partial_x^s,\mathbb{A}]V + \partial_x^s\mathbb{H}\,.$$

By considering $[\partial_x^s,\mathbb{A}]V$ as part of the source term and by using the Duhamel formula, we get

$$|V|_{H^s} \leq C|\mathbb{H}|_{H^s}.$$

This yields

$$|u_1|_s \leq C|F|_{l_k+1}.$$

This ends the proof of Lemma 4.2.

4.7. **Proof of Lemma 4.3.** We study again the equation

$$\sigma w = J(ik)Lw + J(ik)S(ik)w + J(ik)F.$$

Again, we can apply $R(\sigma,k)$ to get

$$\sigma R(\sigma,k)w - R(\sigma,k)\big(J(ik)Lw + J(ik)S(ik)w\big) = R(\sigma,k)J(ik)F. \tag{4.15}$$

To solve (4.15), we use a method close to the one used in [20] in a different context. The problem is that in estimates (4.13), (4.14), we have that $\alpha\approx\rho(k,K)$ may degenerate for $k\sim 0$. The convolution estimate

$$\|e^{-\alpha|x|}\star f(x)\|_{L^2} \leq \frac{C}{|\alpha|}\|f\|_{L^2}$$

gives the rate of degeneration. The strategy is to write the solution w as a sum of two pieces. The first piece satisfies the needed estimate thanks to the 1d assumption (hence no degeneration in the limit $k\to 0$), while the second piece satisfies an equation of type (4.15) with a source term vanishing as $|\mathrm{Re}\ \mu(k,\sigma)|$ in the limit $k\to 0$. This exactly compensates the singularity in the convolution estimate.

To be more precise, we seek the solution of (4.15) under the form

$$w = u + v \tag{4.16}$$

where u solves

$$\sigma R(\sigma, 0^+)u - R(\sigma, 0^+)J(0)Lu = R(\sigma, 0^+)J(0)F \tag{4.17}$$

and hence v solves

$$\begin{aligned}\sigma R(\sigma,k)v - R(\sigma,k)\big(J(ik)Lv + J(ik)S(ik)v\big) = \\ -\Big(R(\sigma,k)J(\sigma,k)S(ik)u + \sigma\big(R(\sigma,k) - R(\sigma,0^+)\big)u \\ +\big(R(\sigma,k)J(ik) - R(\sigma,0^+)J(0)\big)Lu\Big) + \big(R(\sigma,k)J(ik) - R(\sigma,0^+)J(0)\big)F := H.\end{aligned} \tag{4.18}$$

The main interest of this manipulation is that the source term of (4.18) now vanishes thanks to (4.8) when $k \to 0$ if $A_\infty(\sigma, k)$ has an eigenvalue of vanishing real part.

To solve (4.17), we can choose u as the solution of

$$\sigma u - J(0)Lu = J(0)F.$$

Since we assume that D does not vanish on $\mathcal{K}$, we can use Lemma 4.2 to get

$$|u|_s \le C|F|_{s+q}. \tag{4.19}$$

Thanks to the assumption (4.8), this implies that the source term in (4.18) satisfies the estimate

$$|H(\sigma,k)|_s \le C\,\rho(k,K)\,|F|_{s+q+q_1} \tag{4.20}$$

for some $q_1 \ge 0$. To study (4.18), we can use the block structure (4.1) to get

$$v_2 = E(\sigma,k)^{-1}\big(P_2(\sigma,k)v_1 + H_2\big), \quad P_1(\sigma,k)v_1 = H_1.$$

Since by assumption the operators E and P_2 have a continuous extension to $\mathcal{K} \times [0, M]$, we get

$$|v_2|_s \le C\Big(|v_1|_{s+m-1} + |F|_{s+l+q+q_1}\Big)$$

uniformly for $(\sigma, k) \in \mathcal{K} \times (0, M]$. Consequently, we only need to study the equation

$$P_1(\sigma,k)v_1 = H_1$$

to get the result. As in the proof of Lemma 4.2, we rewrite this equation as a first order system

$$V_x = A(\sigma,k,x)V + \mathbb{H}\,. \tag{4.21}$$

To get the existence of exponential dichotomies for

$$V_x = A(\sigma,k,x)V \tag{4.22}$$

on $\mathbb{R}_+$ and $\mathbb{R}_-$ when $k \neq 0$ with a good control of C and α, we can use the conjugation Lemma of [23]. Thanks to Lemma 2.6 of [23], there exist conjugators $\mathcal{W}_\pm(x, \sigma, k)$ such that $\mathcal{W}_\pm(x, \sigma, k)$ are invertible for every (σ, k) with (σ, k) with Re $\sigma > 0$, $k \in [0, M]$ and $x \in \mathbb{R}_\pm$ with a uniform bound of $\mathcal{W}_\pm$ and $\mathcal{W}_\pm^{-1}$ and the property

$$\mathcal{W}_\pm = Id + \mathcal{O}(e^{-\pm\alpha x})$$

when x tends to $\pm\infty$. Moreover, for every V solution of (4.22), $V_1 = \mathcal{W}_\pm{}^{-1} V$ solves

$$(V_1)_x = \mathbb{A}_\infty(\sigma, k) V_1. \tag{4.23}$$

Since for $k \neq 0$, the spectrum of $A_\infty(\sigma, k)$ does not intersect the imaginary axis, the autonomous system (4.23) has an exponential dichotomy on $\mathbb{R}$, for $k \neq 0$. Namely, there exists $P_\infty(\sigma, k)$ and $C > 0$ such that

$$|e^{xA_\infty(\sigma,k)} P_\infty U| \leq Ce^{-\alpha(k)x}|U|, \forall x \geq 0, \forall U \in \mathbb{C}^N \tag{4.24}$$

$$|e^{xA_\infty(\sigma,k)}(I - P_\infty)U| \leq Ce^{\alpha(k)x}|U|, \forall x \leq 0, \forall U \in \mathbb{C}^N . \tag{4.25}$$

where we can take $\alpha(k) = \rho(k, K)/2$. Moreover, P_∞ can be continued up to $k = 0$. Thanks to the conjugation property, we have

$$T(\sigma, k, x) = \mathcal{W}_\pm(\sigma, k, x) e^{xA_\infty(\sigma,k)} \mathcal{W}_\pm(\sigma, k, 0)^{-1}, \; x \in \mathbb{R}_\pm \tag{4.26}$$

and hence the projections $P_\pm(\sigma, k, 0)$ which define the exponential dichotomy for (4.22) are given by

$$P_+(\sigma, k, 0) = \mathcal{W}_+(\sigma, k, 0) P_\infty \mathcal{W}_+(\sigma, k, 0)^{-1}, \quad P_-(\sigma, k, 0) = \mathcal{W}_-(\sigma, k, 0)(Id - P_\infty)\mathcal{W}_-(\sigma, k, 0)^{-1}.$$

Since by assumption, the Evans function $\tilde{D}$ does not vanish up to $k = 0$, we still have that

$$\mathcal{R}P_+(\sigma, k, 0) \oplus \mathcal{R}P_-(\sigma, k, 0) = \mathbb{C}^N.$$

Thanks to (4.24), (4.25) and (4.26), we thus get that (4.13), (4.14) are still true for $\sigma \in \mathcal{K}$ and $|k| \leq M$, $k \neq 0$ with C independent of k and $\alpha = \alpha(k)$.

By using again Duhamel formula and convolution estimates, we get for the solution of (4.21)

$$|V|_s \leq \frac{C}{\alpha(k)} |\mathbb{H}|_s$$

and hence, we can use (4.20) to get

$$|v|_s \leq \frac{C}{\alpha(k)} |H|_s \leq \frac{C\rho(k, K)}{\alpha(k)} |F|_{s+q+q_1} = C|F|_{s+q+q_1}$$

for $k \neq 0$. This ends the proof.

5. Criterion for the existence of multipliers

In this section we prove a criterion for the assumption of Section 2.3.4.

Lemma 5.1. *Suppose that for every $s \geq 2$ there exists a symmetric operator K_s, bounded on L^2 such that*

$$E_s \equiv -\frac{1}{2}\partial_x[J(ik), R]\partial_x - \frac{s}{2}\big(\partial_x J(ik)\,[\partial_x, R] + [\partial_x, R]^* J(ik)\partial_x\big) + \frac{1}{2}[K_s, J(ik)L_0]$$

is an operator of order 1*, i.e. there exists $C_s(k) > 0$ with $|E_s u| \leq C_s(k)|u|_1$. Then, we have that there exists M_s such that (2.17) and (2.18) hold.*

This general criterion can be used in a very simple way when $J(ik)$ is a zero order operator, i.e. $J(ik) \in \mathcal{B}(L^2)$. Indeed, we notice that in such a situation the second term in E_s is already a first order operator. We will prove the following corollary:

Corollary 5.2. *Assume that $J(ik) \in \mathcal{B}(L^2)$ and that $L_0 = -\partial_x^2 + \tilde{L}$ with $\tilde{L} \in \mathcal{B}(H^1, L^2)$. Then $K_s = R$ verifies the assumption of Lemma 5.1 i.e. E_s is a first order operator and hence there exists M_s such that* (2.17)*,* (2.18) *hold.*

5.1. **Proof of Corollary 5.2.** We check that the assumption of Lemma 5.1 is verified with $K_s = R$. As already noticed the second term in the definition of E_s in Lemma 5.1 is already a first order operator since J is a zero order operator. Next, by the assumption $L_0 = -\partial_x^2 + \tilde{L}$, we notice that

$$[K_s, J(ik)L_0] = -\partial_x[K_s, J(ik)]\partial_x + \tilde{E}$$

with $\tilde{E}$ a first order operator. This proves that E_s is indeed a first order operator with the choice $K_s = R$.

5.2. **Proof of Lemma 5.1.** For $s \geq 2$, we define the symmetric operator

$$M_s u \equiv (-1)^s \partial_x^{2s} u + (-1)^{s-1}\partial_x^{s-1}\big(K_s \partial_x^{s-1} u\big).$$

Thanks to the L^2 boundedness of K_s the assumption (2.17) is clearly satisfied. Let us next check (2.18). For that purpose, we need to evaluate the quantity

$$\mathrm{Re}\big((J(ik)(L + S(ik))u, M_s u).$$

Since $J(ik)L_0$ is skew-symmetric, we have

$$\mathrm{Re}\big(J(ik)L_0 u, (-1)^s \partial_x^{2s} u\big) = \mathrm{Re}\big(J(ik)L_0 \partial_x^s u, \partial_x^s u\big) = 0. \tag{5.1}$$

We can also write

$$\mathrm{Re}\big(J(ik)R\,u, (-1)^s \partial_x^{2s} u\big) = \mathrm{Re}\big(J(ik)R\,\partial_x^s u, \partial_x^s u\big) + s\mathrm{Re}\big(J(ik)[\partial_x, R]\partial_x^{s-1} u, \partial_x^s u\big) + (\mathcal{C}u, \partial_x^s u)$$

where $|\mathcal{C}u| \leq C|u|_{s-1}$. Furthermore, since $J(ik)$ is skew symmetric and R symmetric,

$$\begin{aligned}\mathrm{Re}\big(J(ik)R\,\partial_x^s u, \partial_x^s u\big) &= \frac{1}{2}\mathrm{Re}\big((J(ik)R - RJ(ik))\partial_x^s u, \partial_x^s u\big)\\ &= -\frac{1}{2}\mathrm{Re}\big(\partial_x\,[J(ik), R]\,\partial_x\,\partial_x^{s-1}u, \partial_x^{s-1}u\big).\end{aligned}$$

Therefore, we finally find

$$\begin{aligned}&\mathrm{Re}\big(J(ik)R\,u, (-1)^s\partial_x^{2s}u\big)\\ &= -\frac{1}{2}\,\mathrm{Re}\big(\partial_x\,[J(ik), R]\,\partial_x\,\partial_x^{s-1}u, \partial_x^{s-1}u\big)\\ &-\frac{s}{2}\Big(\big(\partial_x\,J(ik)\,[\partial_x, R] + [\partial_x, R]^*\,J(ik)\,\partial_x\big)\partial_x^{s-1}u, \partial_x^{s-1}u\Big) + \mathcal{O}(1)|u|_s\,|u|_{s-1}.\end{aligned} \tag{5.2}$$

Next, since $J(ik)S(ik)$ is skew-symmetric,

$$\mathrm{Re}\big(J(ik)S(ik)u, (-1)^s\partial_x^{2s}u\big) = \big(J(ik)S(ik)\partial_x^s u, \partial_x^s u\big) = 0. \tag{5.3}$$

Since $J(ik)L_0$ is skew-symmetric and K_s symmetric, we also have

$$\begin{aligned}\mathrm{Re}\big(J(ik)L_0 u, (-1)^{s-1}\partial_x^{s-1}\,K_s\,\partial_x^{s-1}\,u\big) &= \mathrm{Re}\big(K_s\,J(ik)L_0\,\partial_x^{s-1}u, \partial_x^{s-1}u\big)\\ &= \frac{1}{2}\big([K_s, J(ik)L_0]\,\partial_x^{s-1}u, \partial_x^{s-1}u\big).\end{aligned} \tag{5.4}$$

Since K_s is bounded on L^2 and $J(ik)$ of order one, we have that

$$\mathrm{Re}\big(J(ik)Ru, (-1)^{s-1}\partial_x^{s-1}\,K_s\,\partial_x^{s-1}\,u\big) = \mathcal{O}(1)|u|_s\,|u|_{s-1}. \tag{5.5}$$

Next, since by the assumptions of section 2.2.2, we have in particular that $J(ik)S(ik)\,\partial_x \in \mathcal{B}(H^2, L^2)$ and since K_s is bounded on L^2, we have

$$\begin{aligned}&\mathrm{Re}\big(J(ik)S(ik)u, (-1)^{s-1}\partial_x^{s-1}K_s\partial_x^{s-1}u\big) =\\ &\qquad \mathrm{Re}\big(J(ik)S(ik)\partial_x\,\partial_x^{s-2}\,u, K_s\partial_x^{s-1}u\big) = \mathcal{O}(1)|u|_s\,|u|_{s-1}.\end{aligned} \tag{5.6}$$

Collecting (5.1), (5.2), (5.3), (5.4), (5.5), (5.6), we infer that

$$\mathrm{Re}\big((J(ik)(L + S(ik))u, M_s u) = (E_s\partial_x^{s-1}u, \partial_x^{s-1}u).$$

In view of the assumption on E_s, we obtain that the assertion of the proof of the lemma holds.

6. Proof of Theorem 1 (periodic perturbations)

The general strategy of the proof is inspired from the work of Grenier [12] in fluid mechanics.

6.1. **Construction of a most unstable eigenmode.** By the assumption there exists an unstable mode with associated transverse frequency $k_0 \neq 0$. The first step of the proof is to find the most unstable eigenmode. This means that we look for an unstable mode with associated transverse frequency mk_0, $m \in \mathbb{Z}$ such that the associated amplification parameter σ has maximal real part. This is indeed possible thanks to the following lemma.

Lemma 6.1. *Consider the problem*

$$\sigma U = J(imk_0)(LU + S(imk_0)U), \quad U \in L^2(\mathbb{R}; \mathbb{C}^d)\,. \tag{6.1}$$

There exists $K > 0$ such that for $|mk_0| \geq K$ there is no nontrivial solution of (6.1) with $Re(\sigma) \neq 0$.
In addition, for every $k \neq 0$ there is at most one unstable mode with corresponding transverse frequency k.

Proof. Recall that by assumption if U solves (6.1) then U belongs to $H^\infty(\mathbb{R}; \mathbb{C}^N) \cap \mathcal{D}_S$. By taking the real part of the scalar product of (6.1) with $LU + S(imk_0)U$, we get the following "conservation law"

$$0 = \mathrm{Re}(\sigma((U, LU) + (U, S(imk_0)U))) = \mathrm{Re}(\sigma)((U, LU) + (U, S(imk_0)U)). \tag{6.2}$$

Indeed, since $J(imk_0)$ is skew-symmetric, we have Re $(J(imk_0)u, u) = 0$ for every $u \in H^\infty$ and we have also used that L and $S(ik)$ are symmetric. Thanks to (2.8), we get that for $|mk_0| \geq K$ there is no nontrivial solution of (6.2) with $\mathrm{Re}(\sigma) \neq 0$.

Let us now prove the second assertion of the lemma, i.e. we shall prove that for $k \neq 0$ there is at most one unstable eigenmode with corresponding transverse frequency k. Thanks to (6.1), we first notice that an unstable eigenmode must be in the image of $J(imk_0)$, consequently, since $J(ik)$ is into, we can write $U = J(ik)V$ with $V \in H^\infty(\mathbb{R}; \mathbb{C}^N)$ a nontrivial solution of

$$\sigma J(ik)V = \big(J(ik)LJ(ik) + J(ik)S(ik)J(ik)\big)V \equiv M_k V, \quad k = mk_0. \tag{6.3}$$

Note that M_k is a symmetric operator. Next, we observe that the operator $J(ik)LJ(ik)$ has at most one positive eigenvalue. Indeed, by contradiction, if $J(ik)LJ(ik)$ had an invariant subspace E of dimension at least 2 on which the quadratic form $J(ik)LJ(ik)$ is positive definite, then the quadratic form (Lu, u) would be negative definite on $J(ik)E$ and since $J(ik)$ is into $J(ik)E$ is also two-dimensional. This gives a contradiction since L has only one simple negative eigenvalue.

Next, we can also prove that M_k has at most one simple positive eigenvalue. Again, if M_k has an invariant subspace E of dimension at least 2 on which the quadratic form $(M_k u, u)$ is positive definite then there exists $u \in E \cap (\psi)^\perp \neq \{0\}$ where ψ is the only positive eigenvalue of $J(ik)LJ(ik)$. Since on $(\psi)^\perp$, $J(ik)LJ(ik)$ is non positive, and $S(ik)$ is positive, we get

$$(M_k u, u) = (J(ik)LJ(ik)u, u) + (J(ik)S(ik)J(ik)u, u) \leq 0$$

which yields a contradiction. Consequently M_k has at most one positive eigenvalue. Finally, we can use [26, Theorem 3.1] to get that $J(ik)^{-1}M_k$ has at most one unstable eigenvalue. Consequently, for $k \neq 0$, there is at most one unstable σ for which (6.1) has a nontrivial solution. This completes the proof of Lemma 6.1. $\square$

By the assumption (2.12), we know that for $k = 0$, there is no unstable eigenmode. We consider the finite set A of integers m such that $k_0 \leq |mk_0| \leq K$, where K is provided by Lemma 6.1. Again by Lemma 6.1, for every $m \in A$ there is at most one unstable mode with corresponding transverse frequency mk_0. Moreover, by the assumption of Theorem 1, for $m = 1$ there is an unstable mode. We now take the unstable mode U corresponding to $m_0 \in A$ with maximal real part of the corresponding amplification parameter which we note by σ_0. We set

$$u^0(t,x,y) \equiv e^{\sigma_0 t} e^{im_0k_0y}\, U + e^{\overline{\sigma_0} t} e^{-im_0k_0y}\, \overline{U} = 2\mathrm{Re}\big(e^{\sigma_0 t} e^{im_0k_0y}\, U\big)\,.$$

To prove Theorem 1, we shall use $Q + \delta u^0(0)$ as an initial data for (2.4). Thanks to our assumptions of section 2.4 about the nonlinear problem, the problem (2.4) is locally well-posed with data $Q + \delta u^0(0)$.

6.2. **Construction of an high order unstable approximate solution.** Denote by $F_j \in C^\infty(\mathbb{R}^d;\mathbb{R})$, $1 \leq j \leq d$ the derivative of F with respect to the j'th variable, i.e. $\nabla F = (F_1, \cdots, F_d)$. For $\alpha \in \mathbb{N}^d$, we set

$$F_\alpha \equiv \big(\partial^\alpha F_1(Q), \cdots, \partial^\alpha F_d(Q)\big). \tag{6.4}$$

Let us look for a solution of (2.4) under the form $u = Q + \delta v$, where $\delta \in]0,1]$. Recall the Taylor formula

$$f(x+y) - f(y) = \sum_{1\leq|\alpha|\leq N} \frac{x^\alpha}{\alpha!}\partial^\alpha f(y) + (N+1)\sum_{|\alpha|=N+1}\frac{x^\alpha}{\alpha!}\int_0^1 (1-t)^N \partial^\alpha f(tx+(1-t)y)dt,$$

where $N \geq 1$ and $f \in C^\infty(\mathbb{R}^d;\mathbb{R})$. In what follows, we shall also use that for $s \geq 2$, $\mathbb{H}^s$ is an algebra , and that $\|f(u)\|_s \leq \Lambda(\|u\|_s)$, where $\Lambda : \mathbb{R} \to \mathbb{R}^+$ is a continuous function. We obtain thus that for every $M \geq 1$, v solves the equation

$$\delta\partial_t v = \mathcal{J}(\partial_y)\Big(\delta(L + \mathcal{S}(\partial_y))v + \sum_{2\leq|\alpha|\leq M+1} \delta^{|\alpha|} v^\alpha F_\alpha + \delta^{M+2} R_{M,\delta}(v)\Big), \tag{6.5}$$

where F_α is defined by (6.4) and $R_{M,\delta}$ satisfies for $s \geq 2$

$$\forall\, \delta \in]0,1], \quad \forall v \in \mathbb{H}^s\,,\ \|R_{M,\delta}(v)\|_s \leq \|v\|_s^{M+2}\Lambda_M(\|\delta v\|_s)\,,$$

where $\Lambda_M : \mathbb{R} \to \mathbb{R}^+$ is a continuous function. We define V_K^s as the space

$$V_K^s = \Big\{ u \,:\, u = \sum_{j=-K}^{K} u_j\, e^{ijm_0k_0y}, \quad u_j \in H^s(\mathbb{R}) \Big\}$$

and we define a norm on V_K^s by $|u|_{V_K^s} = \sup_j |u_j|_s$. Let us notice that u^0 is such that $u^0 \in V_1^s$ for all $s \in \mathbb{N}$. Following the strategy of [12], for $s \gg 1$, we look for an high order solution under the form

$$u^{ap} = \delta u^0 + \sum_{k=2}^{M+1} \delta^k u^k, \quad u^k \in V_{k+1}^s \tag{6.6}$$

such that $u^k_{/t=0} = 0$ and $M \geq 1$ is to be fixed later.

By plugging the expansion in (6.5) and by cancelling the terms involving δ^{k+1}, $1 \leq k \leq M$, we choose u^k so that u^k solves the problem

$$\partial_t u^k = \mathcal{J}(\partial_y)(Lu^k + \mathcal{S}(\partial_y)u^k) + \mathcal{J}(\partial_y) \sum_{2\leq|\alpha|\leq k+1} \Big(\sum_{|\beta|=k+1-|\alpha|} u_1^{\beta_1} \cdots u_d^{\beta_d} \Big) F_\alpha \tag{6.7}$$

where $u_j^{\beta_j}$ stands for the j'th coordinate of u^{β_j} and with the initial condition $u^k_{/t=0} = 0$. Note that the term involving δ cancels thanks to the choice of u^0 (while the term in front of δ^0 is absent in (6.5) thanks to the choice of Q). Thanks to our assumptions u^k is a solutions of a linear equation which is globally defined. Indeed, we can define $\exp(JL_0)$ via the Fourier transform and then treat the problem for u^k perturbatively. Moreover $u^k \in V_{k+1}^s$ for every $s \in \mathbb{R}$. The main point in the analysis of u^{ap} is the following estimate.

Proposition 6.2. *Let us fix an integer $M \geq 1$. Let u^k be the solution of (6.7), $0 \leq k \leq M$. Then for every integer $s \geq 1$ there exists a constant $C_{M,s}$ such that we have the bound*

$$|u^k(t)|_{V_{k+1}^s} \leq C_{M,s} e^{(k+1)Re(\sigma_0)t}, \quad \forall\, t \geq 0. \tag{6.8}$$

As a consequence there exists $G \in \mathbb{H}^s$ for all s such that

$$\partial_t(Q + u^{ap}) - \mathcal{J}(\partial_y)\big(L_0(Q + u^{ap}) + \nabla F(Q + u^{ap}) + \mathcal{S}(\partial_y)(Q + u^{ap})\big) = \mathcal{J}(\partial_y)G$$

and for $0 \leq t \leq \log(1/\delta)/Re(\sigma_0)$ and $s \geq 0$ one has the bound

$$\|\mathcal{J}(\partial_y)G(t)\|_s \leq C_{M,s}\delta^{M+2}\, e^{(M+2)Re(\sigma_0)t},$$

where $C_{M,s}$ is independent of $t \in [0, \log(1/\delta)/Re\ \sigma_0]$ and $\delta \in]0, 1]$.

By an easy induction argument Proposition 6.2 is a consequence of the following statement.

Proposition 6.3. *There exists $q \in \mathbb{N}$ such that for $s \geq 3$, if $f(t) \in V_K^{s+q}$ satisfies*

$$|f(t)|_{V_K^{s+q}} \leq C_{K,s} e^{\gamma t}, \quad \gamma \geq 2Re(\sigma_0) \tag{6.9}$$

then the solution u of the linear problem

$$\partial_t u = \mathcal{J}(\partial_y)(Lu + \mathcal{S}(\partial_y)u) + \mathcal{J}(\partial_y)f, \quad u_{/t=0} = 0. \tag{6.10}$$

belongs to V_K^s and satisfies the estimate

$$|u(t)|_{V_K^s} \leq \tilde{C}_{K,s} e^{\gamma t}, \quad \forall t \geq 0. \tag{6.11}$$

Since f has a finite number of Fourier modes in y, Proposition 6.3 is a direct consequence of the following $1d$ result.

Proposition 6.4. *There exists $q \in \mathbb{N}$ such that for j such that $j/(m_0 k_0) \in \mathbb{Z}$ with $|j|/(m_0 k_0) \leq K$ (where K is provided by Lemma 6.1) and $s \geq 3$, if we suppose also that*

$$|f_j(t)|_{s+q} \leq C_{j,s} e^{\gamma t}, \quad \gamma \geq 2Re(\sigma_0) \tag{6.12}$$

then the solution of

$$\partial_t v = J(ij)\big(L + S(ij)\big)v + J(ij)f_j, \quad v_{/t=0} = 0 \tag{6.13}$$

satisfies

$$|v(t)|_s \leq C_{j,s} e^{\gamma t}.$$

We shall prove below that Proposition 6.4 is a consequence of the following key resolvent estimate.

Proposition 6.5 (Resolvent Estimates). *Let γ_0 be such that $Re(\sigma_0) < \gamma_0 < \gamma$. Suppose that w solves the resolvent equation*

$$(\gamma_0 + i\tau)w = J(ij)\big(L + S(ij)\big)w + J(ij)H \tag{6.14}$$

with $|j|/(m_0 k_0) \leq K$. Then there exists $q \in \mathbb{N}$ such that for $s \geq 1$ an integer there exists $C(s, \gamma_0, K) > 0$ such that for every τ, we have the estimate

$$|w(\tau)|_s \leq C(s, \gamma_0, K)|H(\tau)|_{s+q}. \tag{6.15}$$

6.2.1. *Proposition 6.5 implies Proposition 6.4.* For $T > 0$, we introduce G such that

$$G = 0,\, t < 0, \quad G = 0,\, t > T, \quad G = f_j,\, t \in [0, T]$$

and we notice that the solution of

$$\partial_t \tilde{v} = J(ij)\big(L + S(ij)\big)\tilde{v} + J(ij)G, \quad \tilde{v}_{/t=0} = 0$$

coincides with v on $[0, T]$. Indeed, $w = \tilde{v} - v$ solves for $t \in [0, T]$ the equation

$$\partial_t w = J(ij)\big(L + S(ij)\big)w, \quad w_{/t=0} = 0.$$

By taking the real part of the scalar product of this equation with w, we get by skew-symmetry of $J(ij)L_0$, $J(ij)S(ij)$ and $J(ij)$ that :

$$\frac{d}{dt}|w|^2 = 2\ \mathrm{Re}\ \big(J(ij)Rw, w\big) = -\big(w, [R, J(ij)]w\big).$$

Consequently, thanks to (2.6), we get

$$\frac{d}{dt}|w|^2 \leq C|w|^2$$

and hence, after integration in time , we find that $w = 0$ on $[0, T]$. It is therefore sufficient to study $\tilde{v}$. Next, we set

$$w(\tau, x) = \mathcal{L}\tilde{v}(\gamma_0 + i\tau), \quad H(\tau, x) = \mathcal{L}G(\gamma_0 + i\tau), \quad (\tau, x) \in \mathbb{R}^2$$

where $\mathcal{L}$ stands for the Laplace transform in time :

$$\mathcal{L}f(\gamma_0 + i\tau) = \int_0^\infty e^{-\gamma_0 t - i\tau\, t} f(t)\, dt.$$

By using Proposition 6.5 and Bessel-Parseval identity, we get that for every $T > 0$,

$$\int_0^T e^{-2\gamma_0 t}|v(t)|_s^2\, dt \leq \int_0^{+\infty} e^{-2\gamma_0 t}|\tilde{v}(t)|_s^2\, dt = C\int_{\mathbb{R}} |w(\tau)|_s^2\, d\tau$$
$$\leq C\int_{\mathbb{R}} |H(\tau)|_{s+q}^2\, d\tau = \int_0^T e^{-2\gamma_0 t}|f_j(t)|_{s+q}^2\, dt$$

and finally thanks to (6.9), we get

$$\int_0^T e^{-2\gamma_0 t}|v(t)|_s^2\, dt \leq C\int_0^T e^{2(\gamma-\gamma_0)t}\, dt \leq Ce^{2(\gamma-\gamma_0)T} \tag{6.16}$$

since γ_0 was fixed such that $\gamma > \gamma_0$. To finish the proof, we shall use a crude H^s estimate for the equation (6.13). By using that $J(ij)L_0$ and $J(ij)S(ij)$ are skew-symmetric together with (6.12), we obtain

$$\begin{aligned}\frac{d}{dt}|v(t)|_s^2 &\leq C\Big(|f_j(t)|_{s+1}^2 + 2\mathrm{Re}\sum_{|\alpha|\leq s}\big(J(ij)\partial_x^\alpha(Rv), \partial_x^\alpha v\big)\Big)\\ &\leq C|v(t)|_{s+1}^2 + Ce^{2\gamma t},\end{aligned}$$

where we have used that $J(ij)$ is an operator of order 1. It is possible to have a better estimate involving only $|v|_s^2$ in the right-hand side, but it is useless here. Next, for $0 < \gamma_0 < \gamma$, we get

$$\frac{d}{dt}\Big(e^{-2\gamma_0 t}|v(t)|_s^2\Big) \leq C\Big(e^{-2\gamma_0 t}|v(t)|_{s+1}^2 + e^{2(\gamma-\gamma_0)t}\Big).$$

Therefore, we can integrate in time and use (6.16) (with $s+1$ instead of s) and the fact that $\gamma > \gamma_0$, to find

$$e^{-2\gamma_0 t}|v(t)|_s^2 \leq C e^{2(\gamma-\gamma_0)t}.$$

Therefore, we have shown that Proposition 6.5 implies Proposition 6.4.

6.3. **Proof of Proposition 6.5.** We shall deal differently with the large and bounded temporal frequencies. Indeed, Proposition 6.5 is a consequence of the following two statements.

Lemma 6.6. *For every $\gamma_0 > 0$ and $K \in \mathbb{N}$, there exists $M > 0$ such that for every $s \geq 1$, there exists $C(s, \gamma_0, K)$ such that for $|\tau| \geq M$, $s \geq 1$, we have the estimate*

$$|w(\tau)|_s^2 \leq C(s, \gamma_0, K)|H(\tau)|_{s+1}^2. \tag{6.17}$$

Lemma 6.7. *There exists $q \in \mathbb{N}$ such that for every γ_0, Re $\sigma_0 < \gamma_0 < \gamma$, $s \geq 1$, $K \in \mathbb{N}$ and $M \geq 0$, there exists $C(s, \gamma_0, K, M)$ such that for $|\tau| \leq M$ and $s \geq 1$, we have the estimate*

$$|w(\tau)|_s^2 \leq C(s, \gamma_0, K, M)|H(\tau)|_{s+q}^2. \tag{6.18}$$

6.3.1. *Proof of Lemma 6.6.* We first prove (6.17) for $s = 1$. Recall that the equation (6.14) reads as follows

$$(\gamma_0 + i\tau)w = J(ij)(L + S(ij))w + J(ij)H. \tag{6.19}$$

By the assumption (2.3), we can define an orthogonal decomposition:

$$w = \alpha\varphi_{-1} + w_0 + w_\perp \tag{6.20}$$

where

$$L\varphi_{-1} = \mu\varphi_{-1},\ \mu < 0,\ Lw_0 = 0,\ (Lw_\perp, w_\perp) \geq c_0|w_\perp|_1^2, \quad c_0 > 0. \tag{6.21}$$

Indeed, to obtain the last estimate, we have used that by the assumption 2.3, we have the lower bound

$$(Lw_\perp, w_\perp) \geq c_0|w_\perp|^2, \tag{6.22}$$

but thanks to the decomposition $L = L_0 + R$, and the lower bound (2.2), we also have

$$(Lw_\perp, w_\perp) \geq C^{-1}|w_\perp|_1^2 - C|w_\perp|^2 \tag{6.23}$$

for some $C > 0$ since R is bounded on L^2. Consequently, we can consider $A(6.22) + (6.23)$ with A such that $Ac_0 > C$ to get the claimed in (6.21) lower bound. We normalize φ_{-1} such that $|\varphi_{-1}| = 1$

Note that by the assumption after (2.3), φ_{-1} and w_0 are smooth. Indeed, w_0 is smooth since the kernel of L is spanned by a finite number of smooth eigenvectors and by expaning w_0 on a smooth basis, we also have that for every $s \geq 1$, there exists C_s such that

$$|w_0|_s \leq C_s|w_0|_2 \leq C_s|w|_1. \tag{6.24}$$

Again, we use the conservation law

$$\gamma_0\big((w, Lw) + (w, S(ij)w)\big) = \mathrm{Re}\big((J(ij)H, (L + S(ij))w)\big). \tag{6.25}$$

Consequently, we can use (6.20), (6.21) to get

$$\gamma_0\big(\mu\,\alpha^2|\varphi_{-1}|_1^2 + c_0|w_\perp|_1^2 + |w|^2_{S(ij)}\big) \leq |(J(ij)H, S(ij)w)| + |(J(ij)H, Lw)|.$$

To estimate the right hand side, we first use (2.7) and the skew-symmetry of J to get

$$|(J(ij)H, S(ij)w)| = |(H, J(ij)S(ij)w)| \leq C(j)|H|\,|w|_{S(ij)}.$$

Next, we notice that thanks to (2.2) and Cauchy-Schwarz, we have

$$|(u, L_0 v)| \leq C|u|_1\,|v|_1 \quad \forall u,\, v. \tag{6.26}$$

This yields thanks to (2.5) :

$$|(J(ij)H, Lw)| \leq |J(ij)H|_1\,|w|_1 \leq C|H|_2\,|w|_1.$$

We have thus proven that

$$\gamma_0\big(\mu\,\alpha^2|\varphi_{-1}|_1^2 + c_0|w_\perp|_1^2 + |w|^2_{S(ij)}\big) \leq C|H|_2\Big(|w|_1 + C(j)|w|_{S(ij)}\Big). \tag{6.27}$$

By using the inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon}\,b^2, \quad \forall\,\varepsilon > 0, \quad \forall\,(a, b) \in \mathbb{R}^2, \tag{6.28}$$

with ε small enough, we can incorporate $|w|_{S(ij)}$ in the left hand side of (6.27) and arrive at

$$|w_\perp|_1^2 + |w|^2_{S(ij)} \leq C\big(|\alpha|^2 + |H|_2^2 + |H|_2\,|w|_1\big). \tag{6.29}$$

In what follows C is a large number which may change from line to line and depends on γ_0 and K but not on τ. The next step is to estimate α and w_0. We use the decomposition (6.20) and take the scalar product of (6.19) with $\alpha\varphi_{-1}$ and with w_0 respectively to get

$$(\gamma_0 + i\tau)|\alpha|^2 = -\alpha\big((w, LJ(ij)(\varphi_{-1})) + (J(ij)S(ij)w, \varphi_{-1}) + (J(ij)H, \varphi_{-1})\big)$$
$$(\gamma_0 + i\tau)|w_0|^2 = -(w, LJ(ij)w_0) + (J(ij)S(ij)w, w_0) + (J(ij)H, w_0)$$

and hence, we can take the modulus and add the two identities to get thanks to (6.24) and (2.7) that

$$(\gamma_0 + |\tau|)(|\alpha|^2 + |w_0|^2) \leq C\big(|\alpha|^2 + |w_0|^2 + |w_\perp|^2 + |w|^2_{S(ij)} + |H|^2\big)$$

which we can rewrite as

$$(\gamma_0 + |\tau| - C)(|\alpha|^2 + |w_0|^2) \leq C\big(|w_\perp|^2 + |w|^2_{S(ij)} + |H|^2\big). \tag{6.30}$$

Combining (6.29) and (6.30), we infer that there exists $M > 0$ such that for $|\tau| \geq M$,

$$|w|_1^2 + |w|^2_{S(ij)} \leq C\big(|H|_2\,|w|_1 + |H|_2^2\big). \tag{6.31}$$

To conclude, we use again the inequality (6.28) and we obtain

$$|w|_1^2 + |w|_{S(ij)}^2 \leq C|H|_2^2. \tag{6.32}$$

This proves (6.17) for $s = 1$. Note that moreover (6.32) gives a control of $|w|_{S(ij)}^2$ which is interesting when $j \neq 0$.

In order to estimate higher order derivatives, we use the operator M_s defined in section 2.3.4. By taking the scalar of product of (6.19) by $M_s u$ and taking the real part, since M_s is self-adjoint, we find

$$\gamma_0(w, M_s w) \leq \mathrm{Re}\big((J(L + S(ij))w, M_s w) + (JH, M_s w)\big)$$

and hence we find thanks to (2.17), (2.18),

$$\gamma_0\big(|w|_s^2 - C|w|_{s-1}^2\big) \leq C|w|_s\,|w|_{s-1} + C|JH|_s|w|_s.$$

This yields by a new use of the Young inequality (6.28)

$$|w|_s^2 \leq C\big(|w|_{s-1}^2 + |JH|_s^2\big)$$

and hence, thanks to the assumption on J, we have

$$|w|_s^2 \leq C\big(|w|_{s-1}^2 + |H|_{s+1}^2\big).$$

An induction argument completes the proof of Lemma 6.6.

6.3.2. *Proof of Lemma 6.7.* The assertion of this lemma is a part of our assumptions. Indeed, for $\sigma = \gamma_0 + i\tau$, $|\tau| \leq M$ and every j, $|j| \leq k$, we have by choice of γ_0 that there is no unstable modes on this line which is equivalent to $D(\sigma, j) \neq 0$. Consequently, the assumption (2.14) gives the result.

The proof of Proposition 6.2 is therefore also completed.

6.4. **Nonlinear instability (end of the proof of Theorem 1).** We look for a solution of (2.4) in the form $u = Q + u^{ap} + w$. Then the problem for w to be solved is

$$\partial_t w = \mathcal{J}(\partial_y)(L_0 w + \nabla F(Q + u^{ap} + w) - \nabla F(Q + u_{ap}) + \mathcal{S}(\partial_y)w) - \mathcal{J}(\partial_y)G$$

with zero initial data, where thanks to Proposition 6.2,

$$\|\mathcal{J}(\partial_y)G(t,\cdot)\|_s \leq C_{M,s}\delta^{M+2}e^{(M+2)\mathrm{Re}(\sigma_0)t}, \tag{6.33}$$

as far as $0 \leq t \leq T^\delta$, where

$$T^\delta \equiv \frac{\log(\kappa/\delta)}{\mathrm{Re}(\sigma_0)}$$

with $\kappa \in]0,1[$ small enough, the smallness restriction on κ to be fixed in this section. Thanks to our nonlinear assumption 2.4 and the structure of u^{ap}, w is defined for small times. Next, since $\mathcal{J}(\partial_y)L_0$ and $\mathcal{J}(\partial_y)\mathcal{S}$ are skew symmetric, w enjoys the energy estimate

$$\frac{d}{dt}\|w(t)\|_s^2 \leq \sum_{|\alpha|\leq s}\int_0^1 \Big(\!\Big(\partial^\alpha\big(\mathcal{J}D\nabla F(Q+u^{ap}(t)+\sigma w(t))\cdot w(t)\big),\partial^\alpha w(t)\Big)\!\Big)\,d\sigma+\|\mathcal{J}G(t)\|_s\,\|w(t)\|_s$$

where $(\!(\cdot,\cdot)\!)$ denotes the $L^2(\mathbb{R}\times\mathbb{T}_a)$ scalar product. Let us define a maximum time T^* such that

$$T^* = \sup\{T\,:\,T\leq T^\delta,\ \text{and}\ \forall\, t\in[0,T], ||w(t)||_s\leq 1\}.$$

(T^* is well-defined since $w(0)=0$). Consequently, we can use (2.20), with $s\geq s_0$ to get

$$\|w(t)\|_s^2 \leq \int_0^t \big(\|\mathcal{J}G(\tau)\|_s\|w(\tau)\|_s + \omega(C+\kappa C_{M,s})\|w(\tau)\|_s^2\big)\,d\tau, \tag{6.34}$$

provided also that $t\leq T^\star$. Combining (6.33) and (6.34), we obtain that for $t\in[0,T^*[$,

$$\|w(t)\|_s^2 \leq \omega(C+\kappa C_{M,s})\int_0^T \|w(\tau)\|_s^2 d\tau + C_{M,s}\delta^{2(M+2)}e^{2(M+2)\mathrm{Re}(\sigma_0)t}\,.$$

We take an integer M large enough so that $2(M+2)\mathrm{Re}(\sigma_0)-\omega(C)\geq 2$. At this place we fix the value of M. We then choose κ small enough so that

$$2(M+2)\mathrm{Re}(\sigma_0)-1 > \omega(C+\kappa C_{M,s})-\omega(C)\,.$$

Such a choice of κ is indeed possible thanks to the continuity assumption on ω. By a bootstrap argument and the Gronwall lemma, we infer that $w(t)$ is defined for $t\in[0,T^\delta]$ and that

$$\sup_{0\leq t\leq T^\delta}\|w(t,\cdot)\|_s \leq C_{M,s}\kappa^{M+2}\,.$$

In particular

$$\|w(T^\delta,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} \leq C_{M,s}\kappa^{M+2}\,. \tag{6.35}$$

Let us denote by Π the projection on the nonzero modes in y. For an arbitrary $w\in\mathcal{F}$ (an $L^2(\mathbb{R})$ function depending only on x) one has $\Pi(w)=0$. On the other hand the first term of u^{ap} satisfies $\Pi(u^0)=u^0$ and therefore using (6.8)

$$\begin{aligned}\|\Pi(u^{ap}(t,\cdot))\|_{L^2} &\geq c_s\delta e^{\mathrm{Re}(\sigma_0)t}-\sum_{k=1}^M \delta^{k+1}\|\Pi(u^k)\|_{L^2}\\ &\geq c_s\delta e^{\mathrm{Re}(\sigma_0)t}-\sum_{k=1}^M C_{k,s}\delta^{k+1}e^{(k+1)\mathrm{Re}(\sigma_0)t}\,.\end{aligned}$$

Therefore for κ small enough one has

$$\|\Pi(u^{ap}(T^\delta,\cdot))\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} \geq \frac{c_s\kappa}{2}\,. \tag{6.36}$$

Using (6.35) and (6.36), we may write that for every $w \in \mathcal{F}$,

$$\begin{aligned}
\|u^\delta(T^\delta,\cdot)-w\|_{L^2} &\geq \|\Pi(u^\delta(T^\delta,\cdot)-w)\|_{L^2}\\
&= \|\Pi(u^\delta(T^\delta,\cdot)-Q(\cdot))\|_{L^2}\\
&= \|\Pi(u^{ap}(T^\delta,\cdot)+w(T^\delta,\cdot))\|_{L^2}\\
&\geq \frac{c_s\kappa}{2} - \|\Pi(w(T^\delta,\cdot))\|_{L^2}\\
&\geq \frac{c_s\kappa}{2} - \|w(T^\delta,\cdot)\|_{L^2}\\
&\geq \frac{c_s\kappa}{2} - C_{M,s}\kappa^{M+2}\,.
\end{aligned}$$

A final restriction on κ may insure that the right hand-side of the last inequality is bounded from below by a fixed positive constant η depending only on s (in particular η is independent of δ). This completes the proof of Theorem 1. □

7. Proof of Theorem 2 (localized perturbations)

The first step is to find the most unstable eigenmode which solves

$$\sigma U = J(ik)(LU + S(ik)U). \tag{7.1}$$

Since now k is in $\mathbb{R}$ it is slightly more complicated. We begin with a few preliminary remarks which allow to reduce the search for unstable eigenmodes in a compact set. Thanks to Lemma 6.1 and (2.8), we already now that unstable eigenmodes must be seek only for $k \in [0, K]$, where K is fixed by assumption (2.8). Moreover, by taking the scalar product of (7.1) by U, and then taking the real part, we get

$$\mathrm{Re}(\sigma)|U|^2 \leq \mathrm{Re}\,\left(J(ik)RU, U\right)$$

since $J(ik)L_0$ and $J(ik)S(ik)$ are skew symmetric. Since $J(ik)$ is also skew symmetric, we also have

$$\mathrm{Re}\,\left(J(ik)RU, U\right) = ([J(ik), R]U, U)$$

and hence, thanks to (2.6), we obtain

$$\mathrm{Re}(\sigma)|U|^2 \leq C|U|^2.$$

Consequently, there is no nontrivial solution for $\mathrm{Re}(\sigma)$ sufficiently large. Next, by using the result of Lemma 6.6 for $H = 0$, we find that for every $\gamma_0 > 0$, there exists $C(\gamma_0, K)$ such that there is no nontrivial unstable solution of (7.1) for $\mathrm{Re}\,\sigma \geq \gamma_0$ and $|\mathrm{Im}\,\sigma| \geq C(\gamma_0, K)$.

Now, let us assume that the unstable eigenmode given by our assumption is such that $\mathrm{Re}\,\sigma = \delta$. Then thanks to the previous remarks, the most unstable eigenmode has to be seek in the compact set

$$(\sigma, k) \in \mathcal{R} \equiv \big\{\delta/2 \leq Re\,\sigma \leq C,\ |Im\,\sigma| \leq C(\delta, K)|,\ |k| \leq K\big\}.$$

Moreover, we have an unstable eigenmode if and only if (σ, k) is a zero of the corresponding extended Evans function $\tilde{D}(\sigma, k)$ which is an analytic function in $\{\mathrm{Re}\,\sigma > 0\} \times \mathbb{R}$. We have already proven that for each k there is at most one zero with $\mathrm{Re}\,\sigma > 0$. By Rouche Theorem, if there exists (σ_0, k_0) with $\mathrm{Re}\,\sigma_0 > 0$ such that $\tilde{D}(\sigma_0, k_0) = 0$, there exists a vicinity of σ_0 and k_0 such that for each k, there is exactly one zero $\sigma = \sigma(k)$ of $\tilde{D}$. Moreover, $k \to \sigma(k)$ is analytic. Indeed, we have the explicit expression

$$\sigma(k) = C \int_\Gamma z \frac{\partial_\sigma \tilde{D}(z, k)}{\tilde{D}(z, k)}\, dz,$$

where Γ is a circle which contains σ_0 and C is a constant and hence the analycity of $\tilde{D}$ gives the analycity of σ. If we define $\Omega = \big\{k,\, \exists\, \sigma,\, \mathrm{Re}\,\sigma > \delta/2,\ \tilde{D}(\sigma, k) = 0\big\}$, this proves in particular that Ω is an open bounded set (and non empty thanks to the assumption of the existence of an unstable mode) of $\mathbb{R}$. One can decompose Ω as $\Omega = \cup_m I_m$ where I_m are disjoint, open and bounded intervals which are the connected components of Ω. On each I_m the above considerations prove that there exists an analytic function $k \to \sigma(k)$ such that $\sigma(k)$ is the only zero of $\tilde{D}$ in $\mathrm{Re}\,\sigma > 0$. We shall prove next that $k \to \mathrm{Re}\,\sigma(k)$ has a continuous extension to $\overline{I_m}$. Indeed, if k_n is a sequence converging to an extremity κ of I_m, since $\sigma(k_n)$ is bounded ($\sigma(k_n) \in \mathcal{R}$), then we can extract a sub-sequence not relabelled such that $\sigma(k_n)$ tends to some σ. Moreover, we also have $\mathrm{Re}\,\sigma \geq \delta/2$, and $\tilde{D}(\sigma, \kappa) = 0$, so σ is the only unstable zero of $\tilde{D}(\cdot, \kappa)$. This allows to get that $\lim_{k \to \kappa, k \in I_m} \sigma(k) = \sigma$ and hence to define a continuous function on $\overline{I_m}$. Finally, we also notice that if $\partial I_m \cap \partial I_{m'} \neq \emptyset$, then the continuations must coincide again thanks to the fact that there is at most one unstable eigenmode. Consequently, we have actually a well-defined continuous function $k \to \sigma(k)$ on $\overline{\Omega}$ which is a compact set. This allows to define the most unstable eigenmode as

$$\sigma_0 = \mathrm{Re}\,\sigma(k_0) = \sup\{\sigma_0(k) = \mathrm{Re}\,\sigma(k), \quad k \in \overline{\Omega}\}.$$

Note that, since we have assumed that there exists an unstable mode, σ_0 is positive and also that $k_0 \neq 0$ thanks to the assumption (2.15). Moreover, $\sigma_0(k)$ is an analytic function in the vicinity of k_0 and hence, there exists $m \geq 2$

$$[\mathrm{Re}(\sigma)]'(k_0) = \cdots = [\mathrm{Re}(\sigma)]^{(m-1)}(k_0) = 0, \quad [\mathrm{Re}(\sigma)]^{(m)}(k_0) \neq 0. \tag{7.2}$$

Let I be an interval containing k_0 which does meet zero. For $k \in I$, let us denote by $U(k)$ the unstable mode corresponding to transverse frequency k and amplification parameter

$\sigma(k)$. Then we set

$$\Phi(t,x,y,k,\sigma(k)) = 2\mathrm{Re}\big(e^{\sigma(k)t}e^{iky}\,U(k)\big),$$

where the dependence of x of Φ is in $U(k)$. Further we define

$$u^0(t,x,y) \equiv \int_I \Phi(t,x,y,k,\sigma(k))dk\,.$$

The function u^0 is the first term of our approximate solution, i.e. it is a solution of

$$(\partial_t - \mathcal{J}(\partial_y)L - \mathcal{J}(\partial_y)\mathcal{S}(\partial_y))u^0 = 0\,.$$

Recall that $\sigma_0 \equiv \mathrm{Re}(\sigma)(k_0)$. Thanks (7.2), we can apply the Laplace method (see e.g. [8, 9]) and obtain that for every $s \geq 0$ there exists $c_s \geq 1$ such that for every $t \geq 0$

$$\frac{1}{c_s}\frac{1}{(1+t)^{\frac{1}{2m}}}e^{\sigma_0 t} \leq \|u^0(t,\cdot)\|_{H^s(\mathbb{R}^2)} \leq \frac{c_s}{(1+t)^{\frac{1}{2m}}}e^{\sigma_0 t}\,. \tag{7.3}$$

As in the previous section, we look for an approximate solution of the form

$$u^{ap} = \delta\Big(u^0 + \sum_{k=1}^{M}\delta^k u^k\Big), \quad u^k \in L^2(\mathbb{R}^2)\,, \tag{7.4}$$

where $\delta \ll 1$ and $M \gg 1$ and u^k, $1 \leq k \leq M$ are solutions of (6.7) with zero initial data. Observe that the Fourier transform of u^k with respect to y is compactly supported. Thus using the Fourier transform in y, the Laplace transform in t, and (7.3), we can deduce as in the proof of Theorem 1 the bounds

$$\|u^k(t,\cdot)\|_s \leq \frac{c_{s,k}}{(1+t)^{\frac{k+1}{2m}}}e^{(k+1)\sigma_0 t} \tag{7.5}$$

from the following resolvent estimate.

Proposition 7.1. *Consider $w(\tau)$ the solution of*

$$(\gamma_0 + i\tau)w = J(ik)(L + S(ik))w + J(ik)H, \quad \sigma_0 < \gamma_0 < 2\sigma_0.$$

Then there exists $q \geq 0$ such that for every integer $s \geq 1$, and every $K > 0$, there exists $C(s,\gamma_0,K)$ such that for every $k \in \mathbb{R}\backslash\{0\}$, $|k| \leq K$, and $\tau \in \mathbb{R}$, we have the estimate

$$|w(\tau)|_s^2 \leq C(s,\gamma_0,K)|H(\tau)|_{s+q}^2. \tag{7.6}$$

Proof of Proposition 7.1. As in the proof of Proposition 6.5, we can split the proof of (7.6) into large $|\tau|$ estimates and bounded $|\tau|$ estimates. The proof of the large $|\tau|$ estimate was already proved in Lemma 6.6. As already noticed, there is no difference between the continuous and discrete cases in k for this estimate. To treat the small $|\tau|$ case, we use the assumptions of Section 2.3.3. By choice of γ_0 and thanks to the assumption (2.15), we can use (2.16) for $\mathcal{K} = \{\gamma_0 + i\tau,\, |\tau| \leq M\}$. ∎

With (7.3) and (7.5) at our disposal, we may complete the instability proof as in the previous section. We choose $T^\delta > 0$ such that

$$\frac{e^{T^\delta\,\sigma_0}}{[1+T^\delta]^{\frac{1}{2m}}} = \frac{\kappa}{\delta},$$

where $\kappa > 0$ is small enough to be fixed. Again, we write the solution of (2.4) in the form $u = Q + u^{ap} + w$ with w solution of

$$\partial_t w = \mathcal{J}(\partial_y)(L_0 w + \nabla F(Q + u^{ap} + w) - \nabla F(Q + u_{ap}) + \mathcal{S}(\partial_y)w) - \mathcal{J}(\partial_y)G$$

with zero initial data. Thanks to (7.5), we have that

$$\|\mathcal{J}G(t,\cdot)\|_s \leq C_{M,s}\delta^{M+2}\frac{e^{(M+2)\mathrm{Re}(\sigma_0)t}}{(1+t)^{\frac{M+2}{2m}}}, \quad 0 \leq t \leq T^\delta\,. \tag{7.7}$$

Then thanks to our assumptions, w enjoys the energy estimate

$$\|w(t)\|_s^2 \leq \int_0^t \big(\|\mathcal{J}G(\tau)\|_s\|w(\tau)\|_s + \omega(C+\kappa C_{M,s})\|w(\tau)\|_s^2\big)\,d\tau, \tag{7.8}$$

provided also that $t \leq T^\delta$ and t small. Let us define T^* by

$$T^* \equiv \sup\{T\,:\,T \leq T^\delta,\ \text{and}\ \forall\, t \in [0,T], \|w(t)\|_s \leq 1\}.$$

(T^* is well-defined since $w(0) = 0$). Thanks to (7.7) and (7.8), we obtain that for $t \in [0, T^*[$,

$$\|w(t)\|_s^2 \leq \omega(C+\kappa C_{M,s})\int_0^T \|w(\tau)\|_s^2 d\tau + C_{M,s}\delta^{2(M+2)}\frac{e^{2(M+2)\mathrm{Re}(\sigma_0)t}}{(1+t)^{\frac{M+2}{m}}}\,.$$

We fix an integer M large enough so that $2(M+2)\mathrm{Re}(\sigma_0) - \omega(C) \geq 2$. We then choose κ small enough so that

$$2(M+2)\mathrm{Re}(\sigma_0) - 1 > \omega(C+\kappa C_{M,s}) - \omega(C)\,.$$

Using the inequality

$$\int_0^t \frac{e^{2(M+2)\sigma_0\tau - \omega(C+\kappa C_{M,s})\tau}}{(1+\tau)^{\frac{M+2}{2m}}}d\tau \leq \frac{\tilde{C}\,e^{2(M+2)\sigma_0 t - \omega(C+\kappa C_{M,s})t}}{(1+t)^{\frac{M+2}{2m}}}\,.$$

a bootstrap argument and the Gronwall lemma, we infer that $w(t)$ is defined for $t \in [0, T^\delta]$ and that

$$\|w(T^\delta,\cdot)\|_s \leq C_{M,s}\kappa^{M+2}\,.$$

In particular

$$\|w(T^\delta,\cdot)\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} \leq C_{M,s}\kappa^{M+2}\,. \tag{7.9}$$

Let I_0 be a neighborhood of zero which does not meet I. Let us fix $\varphi \in C_0^\infty(\mathbb{R})$ vanishing on I_0 and equal to one on I. Let us denote by Π the map acting on $\mathcal{S}'(\mathbb{R}^2)$, defined via the Fourier transform as

$$\widehat{\Pi(u)}(\xi_1, \xi_2) = \varphi(\xi_2)\hat{u}(\xi_1, \xi_2).$$

The Fourier multiplier Π is bounded on $L^2(\mathbb{R}^2)$ and we notice that the first term of u^{ap} satisfies $\Pi(u^0) = u^0$. Therefore, we have that

$$\begin{aligned} \|\Pi(u^{ap}(t,\cdot))\|_{L^2} &\geq c_s\delta \frac{e^{\mathrm{Re}(\sigma_0)t}}{(1+t)^{\frac{1}{2m}}} - \sum_{k=1}^{M} \delta^{k+1} \|\Pi(u^k)\|_{L^2} \\ &\geq c_s\delta \frac{e^{\mathrm{Re}(\sigma_0)t}}{(1+t)^{\frac{1}{2m}}} - \sum_{k=1}^{M} C_{k,s}\delta^{k+1} \frac{e^{(k+1)\mathrm{Re}(\sigma_0)t}}{(1+t)^{\frac{k+1}{2m}}}\,. \end{aligned}$$

Therefore for κ small enough one has

$$\|\Pi(u^{ap}(T^\delta, \cdot))\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} \geq \frac{c_s\kappa}{2}\,. \tag{7.10}$$

Finally, since for $w \in \mathcal{F} \subset \mathcal{S}'(\mathbb{R}^2)$, where $\mathcal{F}$ which is defined in statement of Theorem 2 is the set of functions (or tempered distributions) which depends only on x, we have that $\Pi(w) = 0$, by using (7.9), (7.10), we can write that for every $w \in \mathcal{F}$,

$$\begin{aligned} \|u^\delta(T^\delta, \cdot) - w\|_{L^2} &\geq \|\Pi(u^\delta(T^\delta, \cdot) - w)\|_{L^2} \\ &= \|\Pi(u^\delta(T^\delta, \cdot) - Q(\cdot))\|_{L^2} \\ &= \|\Pi(u^{ap}(T^\delta, \cdot) + w(T^\delta, \cdot))\|_{L^2} \\ &\geq \frac{c_s\kappa}{2} - \|\Pi(w(T^\delta, \cdot))\|_{L^2} \\ &\geq \frac{c_s\kappa}{2} - \|w(T^\delta, \cdot)\|_{L^2} \\ &\geq \frac{c_s\kappa}{2} - C_{M,s}\kappa^{M+2}\,. \end{aligned}$$

A final restriction on κ may insure that the right hand-side of the last inequality is bounded from below by a fixed positive constant η. This completes the proof of Theorem 2. $\square$

8. Examples

In this section we give a number of examples when our general result of Theorem 1 and Theorem 2 applies with an unstable mode generated by Lemma 3.1.

8.1. **The generalized KP-I equation.** The $1d$ model is the gKdV equation

$$u_t = \partial_x(-\partial_x^2 - u^p), \quad p = 2, 3, 4, \quad u : \mathbb{R} \to \mathbb{R}. \tag{8.1}$$

For simplicity, we consider only the case of power nonlinearities but more general nonlinear interactions may be considered too. Equation (8.1) has a solution of the form $u(t, x) = Q(x - t)$ with Q smooth with exponential decay. We even have an explicit formula for Q namely

$$Q(x) = \Big(\frac{p+1}{2}\Big)^{\frac{1}{p-1}} \Big(\operatorname{sech}^2(\frac{(p-1)x}{2})\Big)^{\frac{1}{p-1}}. \tag{8.2}$$

The solution $u(t, x) = Q(x-t)$ describes the displacement of the profile Q from left to the right with speed one. One also has the solution

$$u_c(t, x) = c^{\frac{1}{p-1}}\, Q(\sqrt{c}(x - ct)), \quad c > 0 \tag{8.3}$$

which correspond to a solitary wave with a positive speed c. We restrict our considerations only to speed one solitary waves since the case of arbitrary speeds can be reduced to speed one by a change of scale because of (8.3).

By changing x into $x - t$, we observe that Q is stationary solution of

$$u_t = \partial_x(-\partial_x^2 u + u - u^p) \tag{8.4}$$

which fits into the framework of section 2.1 with $d = 1$,

$$J = \partial_x, \quad L_0 = -\partial_x^2 + \mathrm{Id}, \quad F(u) = -\frac{u^{p+1}}{p+1}.$$

Obviously, the first assumptions of section 2.1 are matched. Moreover, we have,

$$L = -\partial_x^2 + \mathrm{Id} - pQ^{p-1}\mathrm{Id}, \quad R = -pQ^{p-1}\mathrm{Id}\,.$$

The spectral condition (2.3) on L is satisfied by Sturm-Liouville theory since Q' has only one zero (see [2, 3]).

The transversally perturbed model is the gKPI equation which reads in the moving frame

$$u_t = \partial_x(-\partial_x^2 u + u - u^p + \partial_x^{-2}\partial_y^2 u). \tag{8.5}$$

Consequently, we have $J(ik) = \partial_x$, $\mathcal{S}(\partial_y) = \partial_x^{-2}\partial_y^2$ and hence $S(ik) = -k^2\partial_x^{-2}$.

The assumptions of section 2.2.1 are obviously met. In particular, since

$$[R, J]w = p(Q^{p-1})_x w,$$

(2.6) is true.

Next, one can also easily check the assumption of section 2.2.2. Note that $|w|^2_{S(ik)} \equiv k^2|\partial_x^{-1}w|^2$, hence, assumption (2.7) is satisfied with $C(k) = |k|$.

Let us next check the assumption (2.8). We have

$$(Lv, v) + (S(ik)v, v) \geq |v_x|^2 + |v|^2 + k^2|\partial_x^{-1}v|^2 - C|v|^2,$$

where $C = p\|Q^{p-1}\|_{L^\infty(\mathbb{R})}$ and hence using the Fourier transform, we find

$$(Lv, v) + (S(ik)v, v) \geq (2\pi)^{-1} \int_{\mathbb{R}} \Big(\xi^2 + \frac{k^2}{\xi^2} + 1 - C\Big)|\hat{v}(\xi)|^2 \, d\xi.$$

and since $\xi^2 + k^2/\xi^2 \geq 2C$ for $|k| \geq C$, we get (2.8) (more precisely for $|k| \geq C$, $\xi^2 + k^2/\xi^2 \geq 2|k| \geq 2C$).

Let us next turn to the assumption on the eigenvalue problem in the context of (8.5). The resolvent equation reads

$$\sigma u = \partial_x(-\partial_x^2 + 1 - pQ^{p-1})u - k^2\partial_x^{-1}u + F_x. \tag{8.6}$$

To prove the existence of the Evans function and (2.14), (2.16), we shall use the criterion of section 4. Let us define

$$R(\sigma, k) = \begin{cases} \partial_x, & \text{if } k \neq 0, \\ \mathrm{Id}, & \text{if } k = 0. \end{cases}$$

then we directly find that $R(\sigma - J(L + S(ik)) = P_1(\sigma, k)$ is a differential operator of order 4 for $k \neq 0$ and 3 for $k = 0$. Consequently, the assumption 4.1 is matched with an empty second block.

For $k \neq 0$, we have (4.4) with

$$A(x, \sigma, k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k^2 - p\partial_x^2(Q^{p-1}) & -\sigma - 2p\partial_x(Q^{p-1}) & 1 - pQ^{p-1} & 0 \end{pmatrix}.$$

Thus

$$A_\infty(\sigma, k) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k^2 & -\sigma & 1 & 0 \end{pmatrix}.$$

The eigenvalues of $A_\infty(\sigma, k)$ are the roots of the polynomial P

$$P(\lambda) = \lambda^4 - \lambda^2 + \sigma\lambda + k^2 \tag{8.7}$$

and hence are not purely imaginary when $\mathrm{Re}\, \sigma > 0$, $k \neq 0$. Moreover, there are two of positive real part and two of negative real part. For $k = 0$, we have

$$A(x, \sigma, 0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sigma - p\partial_x(Q^{p-1}) & 1 - pQ^{p-1} & 0 \end{pmatrix}$$

and thus

$$A_\infty(\sigma,0)=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -\sigma & 1 & 0\end{pmatrix}.$$

The characteristic polynomial of $A_\infty(\sigma,0)$ is $p(\lambda)=-\lambda^3+\lambda+\sigma$ and thus for $\mathrm{Re}(\sigma)>0$ the eigenvalues of $A_\infty(\sigma,0)$ do not meet the imaginary axis.

Consequently, the existence of the Evans function follows from Lemma 4.1.

Finally, since the KdV solitary wave is stable (see e.g. [26]), we have $D(\sigma,0)\neq 0$ when $\mathrm{Re}\,\sigma>0$ and hence the assumption (2.12) is met. Consequently, (2.14) follows from (4.2)

To handle the localized case, we note that when k tends to zero, there is a single root $\lambda=0$ of (8.7) on the imaginary axis and hence, there is spectrum of $A_\infty(\sigma,0^+)$ on the imaginary axis. More precisely, for $k\sim 0$ this root behaves as

$$\mu(\sigma,k)\sim-\frac{k^2}{\sigma}. \tag{8.8}$$

Consequently, there is only one of the negative real part roots of (8.7) which goes to zero. Since $\mu(\sigma,k)$ is analytic, we can use the Gap lemma [11], [17] to get the continuation of the Evans function. Moreover, for k close to zero, we can write the Evans function as

$$\tilde D(\sigma,k)=\begin{vmatrix} \varphi_1^-(0,\sigma,k) & \varphi_2^-(0,\sigma,k) & \varphi_1^+(0,\sigma,k) & \varphi_2^+(0,\sigma,k)\\ \partial_x\varphi_1^-(0,\sigma,k) & \partial_x\varphi_2^-(0,\sigma,k) & \partial_x\varphi_1^+(0,\sigma,k) & \partial_x\varphi_2^+(0,\sigma,k)\\ \partial_x^2\varphi_1^-(0,\sigma,k) & \partial_x^2\varphi_2^-(0,\sigma,k) & \partial_x^2\varphi_1^+(0,\sigma,k) & \partial_x^2\varphi_2^+(0,\sigma,k)\\ \partial_x^3\varphi_1^-(0,\sigma,k) & \partial_x^3\varphi_2^-(0,\sigma,k) & \partial_x^3\varphi_1^+(0,\sigma,k) & \partial_x^3\varphi_2^+(0,\sigma,k)\end{vmatrix},$$

where $\varphi_i^\pm(x,\sigma,k)$, $i=1,2$ decay when x goes to $\pm\infty$ for $k\neq 0$. When $k=0$, φ_i^-, $i=1,2$ and φ_1^+ keep this property as $\varphi_2^+(x,\sigma,0)=c+\mathcal{O}(e^{-\alpha|x|})$, where $c\neq 0$ and $\alpha>0$ are some fixed constants. Note that $\varphi_i^-(x,\sigma,0)$, $i=1,2$ and $\varphi_1^+(x,\sigma,0)$ actually solve

$$\sigma u=\partial_x(-\partial_x^2+1-pQ^{p-1})u \tag{8.9}$$

which is the linearized KdV equation about the solitary wave whereas after integration, we get that $\varphi_2^+(x,\sigma,0)$ solves

$$\sigma u=\partial_x(-\partial_x^2+1-pQ^{p-1})u+c\sigma, \tag{8.10}$$

where the source term $c\sigma$ is identified by looking at the value at ∞ of

$$\sigma\varphi_2^+(x,\sigma,0)-\partial_x(-\partial_x^2+1-pQ^{p-1})\varphi_2^+(x,\sigma,0).$$

Consequently, using (8.9), (8.10) we can write the forth derivatives of $\varphi_i^\pm(0,\sigma,0)$, $i=1,2$ as the same linear combinations of lower order derivatives with an additional term $c\sigma$ for

$\varphi_2^+(0,\sigma,0)$. Therefore, we can perform an operation on the line of the determinant which defines the Evans function, to get that

$$\tilde{D}(\sigma,0) = \begin{vmatrix} \varphi_1^-(0,\sigma,0) & \varphi_2^-(0,\sigma,0) & \varphi_1^+(0,\sigma,0) & \varphi_2^+(0,\sigma,0) \\ \partial_x\varphi_1^-(0,\sigma,0) & \partial_x\varphi_2^-(0,\sigma,0) & \partial_x\varphi_1^+(0,\sigma,0) & \partial_x\varphi_2^+(0,\sigma,0) \\ \partial_x^2\varphi_1^-(0,\sigma,0) & \partial_x^2\varphi_2^-(0,\sigma,0) & \partial_x^2\varphi_1^+(0,\sigma,0) & \partial_x^2\varphi_2^+(0,\sigma,0) \\ 0 & 0 & 0 & c\sigma \end{vmatrix}.$$

Consequently, we get that $|\tilde{D}(\sigma,0)| = |c\sigma D(\sigma,0)|$ where $D(\sigma,0)$ is the Evans function associated to the linearized KdV equation about the solitary wave. Again, since the KdV solitary wave is stable (see e.g. [26]), we also have $\tilde{D}(\sigma,0)$ does not vanish for $\operatorname{Re}\sigma > 0$ and hence, (2.15) is verified. Finally, (4.8) is also met in view of (8.8) since

$$R(\sigma,k)J(ik)S(ik) = \partial_{xx}(-k^2\partial_x^{-2}) = -k^2.$$

Therefore, (2.16) follows from Lemma 4.3.

The assumptions of section 2.3.4 on the existence of a multiplier M_s are also matched. Indeed, we can use the criterion given by Lemma 5.1. Let us set

$$K_s w = r_s(x)\, w$$

where r_s is a smooth and real valued function. A few computation give

$$\begin{aligned} E_s u &= \frac{1}{2}\partial_x\big((pQ^{p-1})_x\partial_x u\big) - \frac{s}{2}\big(-(pQ^{p-1})_x\partial_{xx}u + \partial_{xx}(-(pQ^{p-1})_x u)\big) - \frac{1}{2}[-\partial_x^3+\partial_x, r_s]u \\ &= \Big(\big(\frac{1}{2}+s\big)(pQ^{p-1})_x + \frac{3}{2}(r_s)_x\Big)\partial_{xx}u + \tilde{E}_s u \end{aligned}$$

where $\tilde{E}_s$ is a first order differential operator. Consequently, with the choice

$$r_s = -\big(\frac{1+2s}{3}\big)pQ^{p-1},$$

the properties (2.17), (2.18) are verified. Notice that a similar argument can be performed each time we deal with a scalar equation, i.e. $d=1$ in our general framework.

The "nonlinear" assumptions in the context of (8.5) are also met. In the context of (8.5), (2.19) becomes

$$\partial_t u = -u_{xxx} + u_x + \partial_x^{-1}u_{yy} - \partial_x\big[(u^a+u)^p - (u^a)^p\big] + \partial_x G, \quad u(0)=0. \tag{8.11}$$

To check (2.20), we have to estimate

$$\int \partial^\alpha\partial_x\big((w+v)^{p-1}\,v\big)\,\partial^\alpha v\,dxdy$$

with $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$, $|\alpha_1|+|\alpha_2|\leq s$. Therefore we need to study

$$\int \partial^\alpha\partial_x\big(w^q\,v^r\big)\,\partial^\alpha v\,dxdy, \quad 0\leq q\leq p-1, \quad q+r=p\,,$$

where w may only be putted in L^∞ (or some of its derivatives). If at least one of the derivatives of $\partial^\alpha \partial_x$ acts on w^q then we can use the Gagliardo-Nirenberg-Moser estimates to get the needed bound. Therefore it remains to study

$$\int w^q \partial^\alpha \partial_x \big(v^r\big)\, \partial^\alpha v \, dxdy = r \int w^q \partial^\alpha \big(v^{r-1}\partial_x v\big)\, \partial^\alpha v\, dxdy$$

We write

$$\begin{aligned} \int w^q \partial^\alpha \big(v^{r-1}\partial_x v\big)\, \partial^\alpha v &= \int w^q\, v^{r-1} \partial_x \partial^\alpha v\, \partial^\alpha v + \int w^q [\partial^\alpha, v^{r-1}] \partial_x v\, \partial^\alpha v \\ &= -\frac{1}{2}\int \partial_x (w^q v^{r-1}) |\partial^\alpha v|^2\, dxdy + \int w^q [\partial^\alpha, v^{r-1}] \partial_x v\, \partial^\alpha v \end{aligned}$$

and hence (2.20) follows by the Sobolev embedding and the classical tame commutator estimate

$$\|[\partial^\beta, f] g\| \leq C_\beta \Big(\|f\|_k\, \|g\|_{L^\infty} + \|\nabla f\|_{L^\infty}\, \|g\|_{k-1} \Big), \quad 1 \leq |\beta| \leq k.$$

As already used in the general framework the estimate (2.20) and the fact that JL_0 and $J\mathcal{S}$ are skew-symmetric allow to get and $\mathbb{H}^s$ energy estimate for (8.11).

To get the well-posedness of (8.11), the procedure is very classical and there are several possibilities to achieve this conclusion. One possibility is to consider a regularized version of (8.11), for example

$$\partial_t u^\varepsilon + \varepsilon \Delta^2 \partial_t u^\varepsilon = -u^\varepsilon_{xxx} + u^\varepsilon_x + \partial_x^{-1} u^\varepsilon_{yy} - \partial_x \big[(u^a + u^\varepsilon)^p - (u^a)^p \big] + \partial_x G, \quad u^\varepsilon(0) = 0, \tag{8.12}$$

where $\varepsilon > 0$ and $\Delta = \partial_x^2 + \partial_y^2$. Thanks to the regularization (which is by no means canonical), we may solve (8.12) for a finite time, independent of $\varepsilon > 0$, by means of the Picard iteration applied to the associated integral equation. We next wish to pass to the limit $\varepsilon \to 0^+$ in u^ε. For that purpose, we need to establish an apriori bound on $\|u^\varepsilon(t,\cdot)\|_{H^s}$ independent of ε. These bounds follow from the fact that the well-chosen perturbation enjoy the same H^s estimate as the one formally obtained for (8.11). Then we pass to the limit $\varepsilon \to 0^+$ thanks to a compactness argument. This establishes the local well-posedness of (8.11).

Finally, let us notice that the sufficient condition of Lemma 3.1 for the existence of an unstable mode applies. Indeed, we have

$$M_k = \partial_x L \partial_x - k^2 Id,$$

therefore, it suffices to show that the self adjoint operator $\partial_x L \partial_x$ on $L^2(\mathbb{R})$, with domain $H^4(\mathbb{R})$, has a unique positive eigenvalue. Note that, thanks to Weyl's theorem the essential spectrum of $\partial_x L \partial_x$ is $]-\infty, 0]$. Therefore on $[0,\infty]$ the spectrum of $\partial_x L\partial_x$ can only contains eigenvalues of finite multiplicity and hence for $k > 0$, M_k is Fredholm with zero index. Since L has a unique negative eigenvalue and the remainder of its spectrum is included in $[0,\infty]$, we obtain that, by analyzing the corresponding quadratic forms, the operator $\partial_x L \partial_x$

cannot have more than one positive direction, i.e. u such that $(u, \partial_x L \partial_x u) > 0$. Let us finally show that a positive direction indeed exists. Let us denote by u_{-1} the L^2 normalized eigenvector of L with corresponding to the negative eigenvalue $-\mu$. Let $\varphi_n \in H^{10}(\mathbb{R})$ be a sequence such that $\partial_x \varphi_n$ converges to u_{-1} in $H^{10}(\mathbb{R})$. Then $(\partial_x L \partial_x \varphi_n, \varphi_n)$ converges to $\mu > 0$. Therefore there exist a positive direction of $\partial_x L \partial_x$ which shows that $\partial_x L \partial_x$ has a positive eigenvalue (recall that on $\mathbb{R}^+$ the spectrum of $\partial_x L \partial_x$ can only contains eigenvalues).

Finally, for k_0^2 the unique positive eigenvalue of $\partial_x L \partial_x$, we have

$$\Big([\frac{d}{dk} M_k]_{k=k_0} \varphi, \varphi \Big) = -2k_0 \neq 0$$

and hence (3.1) is verifed. Thus Lemma 3.1 applies in the context of (8.5).

Therefore we can apply our general theory and obtain that Q is (orbitally) unstable as a solution of (8.5) (posed on $\mathbb{R} \times \mathbb{T}_a$ with a suitable a or $\mathbb{R}^2$) thanks to our general results. We have the following statement.

Theorem 8.1. *For every $s \geq 0$, there exists $\eta > 0$ such that for every $\delta > 0$ there exists u_0^δ and a time $T^\delta \sim |\log \delta|$ such that $\|u_0^\delta - Q\|_{H^s(\mathbb{R}^2)} < \delta$ and the generalized KP-I equation (8.5) with data u_0^δ is locally well-posed on $[0, T^\delta]$. Moreover, if we denote by $u^\delta(t)$, $t \in [0, T^\delta]$, the corresponding solution, then $u^\delta(t) - Q \in H^s(\mathbb{R}^2)$ for every $t \in [0, T^\delta]$ and*

$$\inf_{v \in \mathcal{F}} \|u^\delta(T^\delta) - v\|_{L^2(\mathbb{R}^2)} \geq \eta,$$

where $\mathcal{F}$ is the space of $L^2(\mathbb{R})$ functions independent of y.

A similar statement may be done for periodic in y solutions with a suitable period depending on the transverse frequency of the unstable mode (see Theorem 1 above).

Let us recall (see [21, 28]) that for $p = 3, 4$ the generalized KP-I equation, posed on $\mathbb{R}^2$ has local smooth solutions blowing up in finite time, i.e. another (stronger) type of instability exists in these cases. This is in sharp contrast with the case $p = 2$, i.e. the "usual" KP-I equation when global smooth solutions exist both in the case of data periodic in y (see [14]) or localised with respect to Q (or zero), see [25].

8.2. **The nonlinear Schrödinger equation.** The $1d$ model is

$$(i\partial_t + \partial_x^2) u + |u|^2 u = 0, \quad u : \mathbb{R} \to \mathbb{C}.$$

This equation has a solitary wave solution of the form $u(t,x) = e^{it} Q(x)$ with Q smooth with exponential decay. More precisely $Q(x) = \sqrt{2}(\mathrm{ch}(x))^{-1}$. Then after changing u in $e^{it} u$, Q becomes a stationary solution of

$$(i\partial_t + \partial_x^2) u - u + |u|^2 u = 0, \quad u : \mathbb{R} \to \mathbb{C}. \tag{8.13}$$

By writing $u = u_1 + iu_2$ with real valued u_1, u_2, we obtain that $U \equiv (u_1, u_2)^t$ solves the equation

$$\partial_t U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Big((-\partial_x^2 + 1)U + \nabla F(U) \Big), \quad F(U) = -\frac{1}{4}(u_1^2 + u_2^2)^2 \tag{8.14}$$

which fits in our framework with

$$d = 2, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L_0 = -\partial_x^2 + \mathrm{Id}\,.$$

The solution Q of (8.13) is orbitally stable (see [6]). This implies the orbital stability of $(Q, 0)^t$ as a solution of (8.14). The operator L in the context of (8.14) is given by

$$L = \begin{pmatrix} -\partial_x^2 + \mathrm{Id} - 3Q^2 & 0 \\ 0 & -\partial_x^2 + \mathrm{Id} - Q^2 \end{pmatrix}.$$

The spectral condition (2.11) on L is satisfied since $-\partial_x^2 + \mathrm{Id} - 3Q^2$ has exactly two simple eigenvalues -3 and 0 with corresponding eigenvectors Q^2 and Q' and continuous spectrum $[1, \infty[$ while $-\partial_x^2 + \mathrm{Id} - Q^2$ has one simple eigenvalue 0 with corresponding eigenfunction Q and continuous spectrum $[1, \infty[$ (see e.g. [30, 31]).

The transversely perturbed model is the 2D NLS equation that we can write

$$\partial_t U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Big((-\partial_x^2 + 1)U + \nabla F(U) - \partial_y^2 U \Big), \quad F(U) = -\frac{1}{4}(u_1^2 + u_2^2)^2, \tag{8.15}$$

i.e. $\mathcal{S}(\partial_y) = -\partial_y^2$ and $S(ik) = k^2\mathrm{Id}$. The assumptions of sections 2.2.1, 2.2.2 are easy to check. Since Q is bounded, assumption (2.8) is also trivially satisfied.

Let us next turn to the assumption on the key eigenvalue problem in the context of (8.15). Again, we shall use the criteria of section 4. This is very simple in this case, since $\sigma - J(L + S(ik))$ is already a differential operator. Consequently, we can take $R(\sigma, k) = \mathrm{Id}$. If we introduce $V = (u_1, u_2, \partial_x u_1, \partial_x u_2)^t \in \mathbb{C}^4$, $\mathbb{F} = (0, 0, F_2, F_1)^t$ we can rewrite the resolvent equation as $V_x = A(x, \sigma, k)V + \mathbb{F}$, where for all k,

$$A(x, \sigma, k) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k^2 + 1 - 3Q^2 & \sigma & 0 & 0 \\ -\sigma & k^2 + 1 - Q^2 & 0 & 0 \end{pmatrix}.$$

Thus

$$A_\infty(\sigma, k) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ k^2 + 1 & \sigma & 0 & 0 \\ -\sigma & k^2 + 1 & 0 & 0 \end{pmatrix}$$

and we see that A and A_∞ are analytic in (σ, k). We then define $D(\sigma, k)$ as the Wronskian associated to $A(x, \sigma, k)$. Thus $\tilde{D}(\sigma, 0) = D(\sigma, 0)$ in the case of the NLS. The eigenvalues of $A_\infty(\sigma, k)$ are the roots of the polynomial P

$$P(\lambda) = (\lambda^2 - k^2 - 1)^2 + \sigma^2. \tag{8.16}$$

Therefore, in the context of (8.15), for *every* $k \in \mathbb{R}$ the spectrum of $A_\infty(\sigma, k)$ does not meet the imaginary axis. Thus the assumption (4.8) is obviously satisfied. Moreover, since $\tilde{D}(\sigma, 0) = D(\sigma, 0)$, (2.12) and (2.15) are met because of the 1D stability of the solitary wave.

Since J is a zero order operator and L_0 has the required form, we can use Corollary 5.2 to get the existence of a multiplier.

The nonlinear assumptions in the context of (8.13) is satisfied thanks to the standard well-posedness argument for the 2D NLS equation

Moreover, since here J is of order zero, the estimate (2.20) follows by the standard Gagliardo-Nirenberg-Moser inequalities.

Finally, the sufficient condition given by Lemma 3.1 for the existence of an unstable mode applies. Indeed, as for the KP equation, we have

$$M_k = JLJ - k^2 Id$$

since

$$JLJ = -\begin{pmatrix} -\partial_x^2 + \mathrm{Id} - Q^2 & 0 \\ 0 & -\partial_x^2 + \mathrm{Id} - 3Q^2 \end{pmatrix}$$

which have a unique positive eigenvalue. The non-degeneracy condition (3.1) is also obviously verified.

Therefore, our general theory applies and we can state the following results.

Theorem 8.2. *For every $s \geq 0$, there exists $\eta > 0$ such that for every $\delta > 0$ there exists u_0^δ and a time $T^\delta \sim |\log \delta|$ such that $\|u_0^\delta - Q\|_{H^s(\mathbb{R}^2)} < \delta$ and the two dimensional NLS equation*

$$(i\partial_t + \partial_x^2 + \partial_y^2)u + |u|^2 u = 0, \quad u : \mathbb{R}^2 \to \mathbb{C}$$

with data u_0^δ is locally well-posed on $[0, T^\delta]$. If we denote by $u^\delta(t)$, $t \in [0, T^\delta]$, the corresponding solution, then we have $u^\delta(t) - Q \in H^s(\mathbb{R}^2)$, $\forall t \in [0, T^\delta]$ and

$$\inf_{v \in \mathcal{F}} \|u^\delta(T^\delta) - v\|_{L^2(\mathbb{R}^2)} \geq \eta,$$

where $\mathcal{F}$ is the space of $L^2(\mathbb{R})$ functions independent of y.

A similar statement may be done for periodic in y solutions with a suitable period depending on the transverse frequency of the unstable mode (see Theorem 1 above).

8.3. **The Boussinesq equation.** Consider the $1d$ Boussinesq equation

$$u_{tt} + (u_{xx} + u^2 - u)_{xx} = 0, \tag{8.17}$$

This equation has a traveling wave solution (see [4]) of the form

$$u(t,x) = q(x - ct) \equiv q \in H^\infty(\mathbb{R}; \mathbb{R}^2), \quad |c| < 1,\, c \neq 0.$$

In addition q has an exponential decay at infinity. Note that we have

$$q(x) = (1 - c^2)\, Q^{KdV}(\sqrt{1 - c^2}\, x)$$

where Q^{KdV} is the solitary wave with unit speed of the KdV equation given by (8.2) (for $p = 2$). Moreover for $|c| \in]1/2, 1[$ this traveling wave is orbitally stable (see [4]).

At first, we shall rewrite (8.17) as a first order equation. Let us define

$$Bu = -u_{xx} + u$$

and B^α as the Fourier multiplier with symbol $(|\xi|^2 + 1)^\alpha$. Note that B^α is a symmetric operator. By using B, we rewrite (8.17) as

$$u_t = \partial_x B^{\frac{1}{2}} v, \quad v_t = \partial_x B^{-\frac{1}{2}} \Big(Bu - u^2 \Big).$$

Changing x into $x - ct$, we get

$$u_t = \partial_x B^{-\frac{1}{2}} \Big(Bv + cB^{\frac{1}{2}} u \Big), \quad v_t = \partial_x B^{-\frac{1}{2}} \Big(Bu + cB^{\frac{1}{2}} v - u^2 \Big). \tag{8.18}$$

With this change of frame, $Q(x) = (q(x), -cB^{-\frac{1}{2}} q(x))$ is a stationary solution of (8.18). By setting $U = (u, v)^t$,

$$J = \begin{pmatrix} 0 & \partial_x B^{-\frac{1}{2}} \\ \partial_x B^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} B & cB^{\frac{1}{2}} \\ cB^{\frac{1}{2}} & B \end{pmatrix}, \quad F(U) = \begin{pmatrix} -u^3/3 \\ 0 \end{pmatrix}$$

we can write (8.18) under the form (2.1). We easily check that the assumptions of section 2.1 are matched. Note that by Bessel identity, we have

$$(L_0 U, U) = (2\pi)^{-1} \int_{\mathbb{R}} \Big((1 + |\xi|^2) |\hat{u}(\xi)|^2 + (1 + |\xi|^2) |\hat{v}(\xi)|^2 + 2c \,(\sqrt{1 + |\xi|^2}\, \hat{u}(\xi)\, \overline{\hat{v}(\xi)} \Big)\, d\xi$$

and hence, (2.2) is verified for $c < 1$. Moreover, an important remark is that J is here a bounded operator, in contrast with the formulation used by [4].

Next, let us check (2.3). The operator L is defined by $L = L_0 + R$, where

$$R = \begin{pmatrix} -2q & 0 \\ 0 & 0 \end{pmatrix}.$$

The spectral condition on L is again satisfied thanks to the Sturm-Liouville theory and is proven for $1/2 < |c| < 1$ in [4] in order to prove the nonlinear stability of the solitary

wave. Note that the formulation (8.18) that we use is equivalent to the one used by Bona and Sachs in [4].

The transversally perturbed model is

$$u_{tt} + (u_{xx} + u^2 - u - \partial_x^{-2}\partial_y^2 u)_{xx} = 0. \tag{8.19}$$

The equation (8.19) has been derived in [15] as a model for interacting shallow water waves. Again, to rewrite this equation as a first order system, we introduce

$$\mathcal{B}w = -w_{xx} + w + \partial_x^{-2} w_{yy}.$$

We write (8.19) as

$$u_t = \partial_x \mathcal{B}^{\frac{1}{2}} v, \quad v_t = \partial_x \mathcal{B}^{-\frac{1}{2}}\Big(\mathcal{B}u - u^2\Big) \tag{8.20}$$

and hence going into the moving frame, we find

$$u_t = \partial_x \mathcal{B}^{-\frac{1}{2}}\Big(\mathcal{B}v + c\mathcal{B}^{\frac{1}{2}}u\Big), \quad v_t = \partial_x \mathcal{B}^{-\frac{1}{2}}\Big(\mathcal{B}u + c\mathcal{B}^{\frac{1}{2}}v - u^2\Big).$$

Consequently, we get a system under the form (2.4) with

$$\mathcal{J}(\partial_y) = \begin{pmatrix} 0 & \partial_x \mathcal{B}^{-\frac{1}{2}} \\ \partial_x \mathcal{B}^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \mathcal{S}(\partial_y) = \begin{pmatrix} \partial_x^{-2}\partial_y^2 & c\big(\mathcal{B}^{\frac{1}{2}} - B^{\frac{1}{2}}\big) \\ c\big(\mathcal{B}^{\frac{1}{2}} - B^{\frac{1}{2}}\big) & \partial_x^{-2}\partial_y^2 \end{pmatrix}.$$

Therefore the $1d$ operators $J(ik)$ and $S(ik)$ are defined as

$$J(ik) = \begin{pmatrix} 0 & \partial_x B(ik)^{-\frac{1}{2}} \\ \partial_x B(ik)^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad S(ik) = \begin{pmatrix} -k^2\partial_x^{-2} & c\big(B(ik)^{\frac{1}{2}} - B^{\frac{1}{2}}\big) \\ c\big(B(ik)^{\frac{1}{2}} - B^{\frac{1}{2}}\big) & -k^2\partial_x^{-2} \end{pmatrix}$$

with

$$B(ik)w = -w_{xx} + w - k^2\partial_x^{-2}.$$

Note that $J(ik)$ is a bounded operator on L^2. Indeed, its symbol is given by

$$\begin{pmatrix} 0 & \dfrac{i\xi}{\big(1+\xi^2+\frac{k^2}{\xi^2}\big)^{\frac{1}{2}}} \\ \dfrac{i\xi}{\big(1+\xi^2+\frac{k^2}{\xi^2}\big)^{\frac{1}{2}}} & 0 \end{pmatrix}.$$

We can easily check that the assumptions of section 2.2.1 are verified. Since J is bounded on L^2, the estimate (2.6) and (2.5) are obvious.

We also easily check the assumptions of section 2.2.2. The first three assumptions can be verified by computation. Moreover, we notice that there exist positive constants c_0, C_0 such that

$$c_0 k^2 |\partial_x^{-1} U|^2 \leq \big(S(ik)U, U\big) \leq C_0 k^2 |\partial_x^{-1}U|^2. \tag{8.21}$$

Indeed by using the Fourier transform, we have

$$\big(S(ik)U,U\big) = (2\pi)^{-1}\int_{\mathbb{R}}\Big(\frac{k^2}{\xi^2}\big(|\hat{u}(\xi)|^2+|\hat{v}(\xi)|^2\big)+2c\frac{\frac{k^2}{\xi^2}}{\big(1+\xi^2+\frac{k^2}{\xi^2}\big)^{\frac{1}{2}}+\big(1+\xi^2\big)^{\frac{1}{2}}}\hat{u}(\xi)\,\overline{\hat{v}(\xi)}\Big)d\xi$$

and hence (8.21) follows since $c<1$. Thanks to (8.21), we also find that (2.7) is verified with $C(k)=k$.

Next, to get (2.8), we use again the Fourier transform to write

$$(LU,U)+(S(ik)U,U)\geq C_1\int_{\mathbb{R}}(1-c)\big(1+\xi^2+\frac{k^2}{\xi^2}-C\big)|\hat{U}(\xi)|^2d\xi,$$

where $C_1\approx|Q|_{L^\infty}$. Consequently, we get (2.8) as for the KP equation.

The assumptions of section 2.3.4 on the existence of suitable multipliers follows again from Corollary 5.2. Indeed, $J(ik)$ is a zero order operator and $L_0=-\partial_x^2+\tilde{L}$ with $\tilde{L}$ a first order operator.

Let us turn to the study of the resolvent equation (2.13). We first notice that

$$\sigma\mathrm{Id}-J(ik)(L+S(ik))=\begin{pmatrix}\sigma-c\partial_x & -\partial_x B(ik)^{\frac{1}{2}}\\ -\partial_x\,B(ik)^{-\frac{1}{2}}\big(B(ik)-2q\big) & \sigma-c\partial_x\end{pmatrix}.$$

Consequently, by using again section 4, we can set

$$R(\sigma,k)=\begin{pmatrix}\sigma-c\partial_x & \partial_x B(ik)^{\frac{1}{2}}\\ 0 & 1\end{pmatrix}$$

to get (4.1) with

$$\begin{aligned}&P_1(\sigma,k)u=\partial_x^4u-\partial_x^2u+k^2u+2\partial_x^2(qu)+(\sigma-c\partial_x)^2u,\\&E(\sigma,k)=\sigma-c\partial_x,\\&P_2(\sigma,k)u=-\partial_xB(ik)^{-\frac{1}{2}}\big(B(ik)u-2qu\big)=-\partial_xB(ik)^{\frac{1}{2}}u-2\partial_xB(ik)^{-\frac{1}{2}}(qu).\end{aligned}$$

Consequently, P_1 is a fourth order differential operator analytic in (k,σ) for every k, E is invertible for $\mathrm{Re}\,\sigma>0$ and P_2 is a second order operator with domain H^2. Indeed, $\partial_xB(ik)^{-\frac{1}{2}}$ is a bounded operator on L^2 and we have the estimate

$$2\pi\|\partial_xB(ik)^{\frac{1}{2}}u\|^2=\int_{\mathbb{R}}\xi^2\big(\xi^2+1+\frac{k^2}{\xi^2}\big)|\hat{u}(\xi)|^2\,d\xi\leq\int_{\mathbb{R}}\big(1+\xi^4+k^2\big)|\hat{u}(\xi)|^2d\xi,$$

which is uniform for k in a vicinity of zero.

One can rewrite (4.3) in the context of the Boussinesq equation as a first order system (4.4) with $A(x,\sigma,k)\in M_4(\mathbb{C})$ for every k. The assumption (4.6) is met since q decays

exponentially fast to zero at infinity. Moreover, the eigenvalues λ of $A_\infty(\sigma, k)$ are the roots of the polynomial P defined as

$$P(\lambda) = \lambda^4 - (1-c^2)\lambda^2 - 2c\sigma\lambda + k^2 + \sigma^2.$$

Suppose that P has a root of the form $\lambda = i\mu$ with $\mu \in \mathbb{R}$. Then by separating the real and the imaginary part of $P(i\mu)$, we get the relations

$$\mu^4 + (1-c^2)\mu^2 + 2c\sigma_2\mu + \sigma_1^2 - \sigma_2^2 + k^2 = 0, \quad -2c\sigma_1\mu + 2\sigma_1\sigma_2 = 0,$$

where $\sigma = \sigma_1 + i\sigma_2$, $\sigma_1, \sigma_2 \in \mathbb{R}$. Therefore, since $\mathrm{Re}(\sigma) = \sigma_1 \neq 0$, we have that $\mu = \sigma_2/c$ (recall that we are interested for the values of c such that $1/2 < |c| < 1$). By substituting the value of μ in the first equation, we get

$$\frac{\sigma_2^4}{c^4} + \frac{\sigma_2^2}{c^2} + \sigma_1^2 + k^2 = 0. \tag{8.22}$$

But since in the last equation for σ_2, if σ_2 is real, all the terms are non-negative and $\sigma_1^2 > 0$, there is no real root for every $k \in \mathbb{R}$. Therefore for every $k \in \mathbb{R}$ and $\mathrm{Re}(\sigma) \neq 0$ the equation $P(\lambda) = 0$ has no root on the imaginary axis. Since for $k = 0$ there is no complication coming from the emergence of a root on the imaginary axis, we are in the same situation as for the nonlinear Schrödinger equation. The assumption (2.12) (and hence also (2.15)) is met since the solitary wave q is stable as a solution of the $1D$ Boussinesq equation for $1/2 < |c| < 1$ as shown in [4], we get (2.14), (2.16) from Lemma 4.2 and Lemma 4.3.

The "nonlinear" assumptions of section 2.4 are also met. Indeed, the local well-posedness of the $2d$ Boussinesq equation which is semi-linear can be obtained by standard techniques. Moreover, since $\mathcal{J}$ is a bounded operator on H^s, the assumption (2.20) follows readily from the Gagliardo-Nirenberg-Moser inequality.

Remark 8.1. Let us observe that if we consider transverse perturbation with the opposite sign that even the problem defining the free evolution is ill-posed in Sobolev spaces. Thus in the context of the Boussinesq equation the analogue of the KP-II equation is not a "good" model in Sobolev spaces. Indeed consider the linear problem

$$u_{tt} + \big(u_{xx} - u + \partial_x^{-2}\partial_y^2 u\big)_{xx} = 0, \tag{8.23}$$

Using the Fourier transform, we obtain that $\hat{u}$ solves

$$\hat{u}_{tt} + \big(\xi^4 + \xi^2 - k^2\big)\hat{u} = 0,$$

and hence, one can find growing modes $e^{\lambda t}\hat{u}(\xi, k)$ if

$$P(\lambda) = \lambda^2 + \xi^4 + \xi^2 - k^2 = 0\,.$$

Since one can find roots of P with arbitrary large real parts and arbitrary sign this implies the ill-posedness of (8.23). A similar phenomenon occurs for the equation

$$u_{tt} + (-u_{xx} - u - \partial_x^{-2}\partial_y^2)_{xx} = 0, \tag{8.24}$$

where the sign is changed in front of the dispersion term. In view of this discussion, it becomes reasonable to study (8.23) or (8.24) in analytic spaces.

We can also use Lemma 3.1 to get an unstable eigenmode. Indeed, we have

$$M_k = \begin{pmatrix} \partial_{xx} & c\partial_{xx}B(ik)^{-\frac{1}{2}} \\ c\partial_{xx}B(ik)^{-\frac{1}{2}} & \partial_x B(ik)^{-\frac{1}{2}}\big(B(ik)-2q\big)\partial_x B(ik)^{-\frac{1}{2}} \end{pmatrix}.$$

Consequently $(u,v)^t \in L^2(\mathbb{R};\mathbb{R}^2)$ is in the kernel of M_k if and only if

$$u = -cB(ik)^{-\frac{1}{2}}v, \quad \partial_x B(ik)^{-\frac{1}{2}}\Big(-c^2+B(ik)-2q\Big)\partial_x B(ik)^{-\frac{1}{2}}v = 0.$$

Next, we notice that

$$\begin{aligned}\partial_x B(ik)^{-\frac{1}{2}}&\Big(-c^2+B(ik)-2q\Big)\partial_x B(ik)^{-\frac{1}{2}} = \\ &= B(ik)^{-\frac{1}{2}}\Big(\partial_x\big(-\partial_x^2+(1-c^2)-2q\big)\partial_x - k^2\Big)B(ik)^{-\frac{1}{2}} \equiv B(ik)^{-\frac{1}{2}}m_k\,B(ik)^{-\frac{1}{2}}.\end{aligned}$$

Notice that m_k is the operator $JLJ+JSJ$ which appears in the study of the stability of the solitary wave with speed $1-c^2$ of the KP-I equation. Thus as in the analysis for the KP-I equation, we can show that m_k has a one-dimensional non trivial kernel for some $k_0 \neq 0$. This implies that M_{k_0} also has a non-trivial kernel generated by

$$\varphi = (-c\psi, B(ik_0)^{\frac{1}{2}}\psi),$$

ψ being nontrivial and such that $m_{k_0}\psi = 0$. Moreover, we can deduce that M_{k_0} is Fredholm index 0 from the fact that m_{k_0} is Fredholm index 0. Let us check the non-degeneracy condition (3.1). Using the identity

$$\frac{d}{dk}_{/k=k_0} B(ik)^{-\frac{1}{2}} = k_0\partial_x^{-2}B(ik_0)^{-\frac{3}{2}}, \tag{8.25}$$

we obtain that

$$\Big(\Big[\frac{d}{dk}M_k\Big]_{k=k_0}\varphi,\varphi\Big) = -2k_0|\psi|^2 \neq 0 \tag{8.26}$$

since $k_0 \neq 0$. More precisely $M_k = M_k^1 + M_k^2$ with

$$M_k^1 = \begin{pmatrix} \partial_{xx} & 0 \\ 0 & \partial_x B(ik)^{-\frac{1}{2}}\big(B(ik)-2q\big)\partial_x B(ik)^{-\frac{1}{2}} \end{pmatrix}$$

and

$$M_k^2 = \begin{pmatrix} 0 & c\partial_{xx}B(ik)^{-\frac{1}{2}} \\ c\partial_{xx}B(ik)^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

A use of (8.25) gives

$$\Big(\Big[\frac{d}{dk}M_k^2\Big]_{k=k_0}\varphi,\varphi\Big) = -2c^2k_0(B(ik_0)^{-1}\psi,\psi)$$

and (using that $m_{k_0}\psi = 0$)

$$\Big(\Big[\frac{d}{dk}M_k^1\Big]_{k=k_0}\varphi,\varphi\Big) = 2c^2k_0(B(ik_0)^{-1}\psi,\psi) - 2k_0|\psi|^2\,.$$

Thus the identity (8.26) indeed holds true. This allows to use Lemma 3.1 to get the existence of an unstable eigenmode.

Consequently, we can apply our general theory to get the following statement.

Theorem 8.3. *Consider the equation* (8.20) *for* $|c| \in (1/2, 2)$. *For every* $s \geq 0$, *there exists* $\eta > 0$ *such that for every* $\delta > 0$ *there exists* u_0^δ *and a time* $T^\delta \sim |\log\delta|$ *such that* $\|u_0^\delta - Q\|_{H^s(\mathbb{R}^2)} < \delta$ *and the solution* $u^\delta(t)$ *of* (8.20) *with data* u_0^δ *is defined on* $[0, T^\delta]$, *with* $u^\delta(t) - Q \in H^s(\mathbb{R}^2)$, $\forall t \in [0, T^\delta]$ *and moreover satisfies the estimate*

$$\inf_{v\in\mathcal{F}} \|u^\delta(T^\delta) - v\|_{L^2(\mathbb{R}^2)} \geq \eta,$$

where $\mathcal{F}$ *is the space of* $L^2(\mathbb{R})$ *functions independent of* y.

A similar statement may be done for periodic in y solutions with a suitable period depending on the transverse frequency of the unstable mode (see Theorem 1 above).

8.4. **The Zakharov-Kuznetsov equation.** The Zakharov-Kuznetsov equation

$$u_t + u_{xxx} + u_{xyy} + uu_x = 0 \tag{8.27}$$

is derived in [32] to describe the propagation of nonlinear ionic-sonic waves in plasma magnetic field. Equation (8.27) is a two dimensional generalization of the KdV equation which fits into our general framework with $d = 1$. Indeed, if we denote by Q the suitable speed one KdV solitary wave, then Q is a stationary solution of

$$u_t - u_x + u_{xxx} + u_{xyy} + uu_x = 0. \tag{8.28}$$

We can write (8.28) as

$$u_t = J(L_0 + \nabla F(u) + \mathcal{S}(\partial_y))u$$

with

$$J = \partial_x, \quad L_0 = -\partial_x^2 + \mathrm{Id}, \quad F(u) = -\frac{u^3}{6}, \quad \mathcal{S}(\partial_y) = -\partial_y^2\,.$$

Assumptions of section 2.1 are still verified since as for the KP-I equation, the 1d model is the KdV equation. Assumptions 2.2.1, 2.2.2, (2.8) are easy to check.

The operator

$$\sigma u - J(L + S(ik))u = \sigma u - u_x + u_{xxx} - k^2u_x + 2(Qu)_x$$

is already a differential operator and hence we can readily use section 4 with $R(\sigma, k) = \mathrm{Id}$. In particular, we have for every k

$$A(x, \sigma, k) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sigma - 2Q_x & (1 + k^2) - 2Q & 0 \end{pmatrix}.$$

This allows to find that the eigenvalues of $A_\infty(\sigma, k)$ are the roots of the polynomial

$$P(\lambda) = \lambda^3 - (1 + k^2)\lambda + \sigma.$$

Since, for $\xi \in \mathbb{R}$, we have $\mathrm{Re}\ P(i\xi) = \mathrm{Re}\ \sigma$, we get that for $\mathrm{Re}\ \sigma \neq 0$ there is no eigenvalue of $A_\infty(\sigma, k)$ on the imaginary axis. As in the case of NLS, the assumptions of section 4.5 are obviously verified since we are in a situation where $A(x, \sigma, k)$ is analytic for every k and where there is no eigenvalue of A_∞ on the imaginary axis even for $k = 0$. Moreover, (2.12) (and hence also (2.15)) are verified thanks to the stability of the KdV solitary wave. Consequently, the assumptions of sections 2.3.2 and 2.3.3 follow from Lemma 4.2 and Lemma 4.3.

To check assumption 2.3.4, we use again Lemma 5.1. By taking

$$K_s u = -\frac{2}{3}(1 + 2s)Q\, u,$$

we find that E_s is a first order operator and hence the assumption of existence of multiplier of section 2.3.4 follows from Lemma 5.1.

The assumptions of section 2.4 i.e about the local well posedness of the nonlinear equation are again verified by standard arguments. Note that assumption (2.20) was already checked in the study of the KP equation.

Finally, we note that it does not seem possible to use the simple criterion of Lemma 3.1 to prove the existence of an unstable eigenmode. Indeed, we have

$$M_k = \partial_x (L - k^2)\partial_x, \quad L = -u_{xx} + u - 2Q.$$

It is easy to prove that there exists k_0 such that M_{k_0} has a non-trivial kernel. Nevertheless, here M_{k_0} is not a Fredholm operator with index zero. Fortunately, the existence of unstable modes was obtained in [5] by using more sophisticated arguments (i.e. the multisymplectic formulation of the equation). Consequently, we have the following result.

Theorem 8.4. *Consider the equation* (8.28). *For every $s \geq 0$, there exists $\eta > 0$ such that for every $\delta > 0$ there exists u_0^δ and a time $T^\delta \sim |\log \delta|$ such that $\|u_0^\delta - Q\|_{H^s(\mathbb{R}^2)} < \delta$ and the solution $u^\delta(t)$ of* (8.28) *with data u_0^δ is defined on $[0, T^\delta]$ with $u^\delta(t) - Q \in H^s(\mathbb{R}^2)$, $\forall t \in [0, T^\delta]$ and moreover satisfies the estimate*

$$\inf_{v \in \mathcal{F}} \|u^\delta(T^\delta) - v\|_{L^2(\mathbb{R}^2)} \geq \eta,$$

where $\mathcal{F}$ is the space of $L^2(\mathbb{R})$ functions independent of y.

A similar statement may be done for periodic in y solutions with a suitable period depending on the transverse frequency of the unstable mode (see Theorem 1 above).

8.5. **KP-BBM.** Consider the generalized BBM equation

$$u_t - u_{txx} + u_x + \partial_x(u^p) = 0 \tag{8.29}$$

and the $2d$ generalization of KP type

$$u_t - u_{txx} + u_x + \partial_x(u^p) - \partial_x^{-1} u_{yy} = 0. \tag{8.30}$$

For $c > 1$ there is a solitary wave solution of (8.29) of the form $u(t,x) = Q(x - ct)$. Again, we note that

$$Q(x) = (c-1)^{\frac{1}{p-1}} Q^{KdV}\big(\sqrt{1 - \frac{1}{c}}x\big).$$

Then $Q(x)$ is a stationary solution of the equation

$$u_t - (c-1)u_x - u_{txx} + cu_{xxx} + \partial_x(u^p) - \partial_x^{-1} u_{yy} = 0. \tag{8.31}$$

Equation (8.31) may be written under the form

$$\partial_t u = J(L_0 + \nabla F(u) + \mathcal{S}(\partial_y))u,$$

where

$$J = (1 - \partial_x^2)^{-1}\partial_x, \quad L_0 = -c\partial_{xx} + (c-1)Id, \quad F(u) = -\frac{1}{p+1}\int u^{p+1}, \quad S(ik)u = -k^2\partial_x^{-2}.$$

The corresponding operator L is

$$L(u) = -cu_{xx} + (c-1)u - pQ^{p-1}u\,.$$

Again, it is very easy to check the assumptions of sections 2.1, 2.2.1, 2.2.2, 2.8. To ensure that (2.11) and (2.12) are verified, we restrict ourself to $p \leq 4$, in this case all the waves for $c > 1$ are stable in the 1D model which is the BBM equation [26], [3].

Note that we are in a semilinear situation since J is a zero order operator (and even better). Consequently, the assumption of existence of multiplier of section 2.3.4 is verified thanks to Corollary 5.2 (recall that for scalar problems it is straightforward). The assumption 2.20 is met thanks to the Gagliardo-Nirenberg-Moser inequality. Again the local well-posedness assumed in section 2.4 can be proven by standard methods.

To check (2.14), (2.16), we can use section 4. Since

$$\sigma u - J\big(L + S(ik)\big)u = \sigma u - (1 - \partial_x^2)^{-1}\partial_x\big(-c\partial_{xx}u + (c-1)u - pQ^{p-1}u - k^2\partial_x^{-2}u\big),$$

we set

$$R(\sigma, k) = \begin{cases} (\mathrm{Id} - \partial_x^2)\partial_x, & \text{if } k \neq 0, \\ \mathrm{Id} - \partial_x^2, & \text{if } k = 0. \end{cases}$$

Then we directly find that $R(\sigma, k) - J(L + S(ik))) = P_1(\sigma, k)$ is a differential operator of order 4 for $k \neq 0$ and 3 for $k = 0$. Consequently, the assumption of section 4.1 is matched with an empty second block.

For $k \neq 0$, we have (4.4) with

$$A(x, \sigma, k) = c^{-1} \begin{pmatrix} 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \\ -k^2 - p\partial_x^2(Q^{p-1}) & -\sigma - 2p\partial_x(Q^{p-1}) & c - 1 - pQ^{p-1} & \sigma \end{pmatrix}.$$

Thus

$$A_\infty(\sigma, k) = c^{-1} \begin{pmatrix} 0 & c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \\ -k^2 & -\sigma & c-1 & \sigma \end{pmatrix}.$$

The eigenvalues of $A_\infty(\sigma, k)$ are the roots of the polynomial P

$$P(\lambda) = c\lambda^4 - \sigma\lambda^3 - (c-1)\lambda^2 + \sigma\lambda + k^2 \tag{8.32}$$

and hence are not purely imaginary when $\operatorname{Re}\sigma > 0$. Moreover, there are two of positive real part and two of negative real part. For $k = 0$, we have

$$A(x, \sigma, 0) = c^{-1} \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & c \\ -\sigma - p\partial_x(Q^{p-1}) & 1 - pQ^{p-1} & 0 \end{pmatrix}$$

and thus

$$A_\infty(\sigma, 0) = c^{-1} \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & c \\ -\sigma & c-1 & \sigma \end{pmatrix}.$$

The characteristic polynomial of $A_\infty(\sigma, 0)$ is $p(\lambda) = c\lambda^3 - \sigma\lambda^2 - (c-1)\lambda + \sigma$ and thus for $\operatorname{Re}(\sigma) > 0$ the eigenvalues of $A_\infty(\sigma, 0)$ do not meet the imaginary axis. This allows to use Lemma 4.1 to get the existence of the Evans function. Finally, since the BBM solitary wave is stable (see e.g. [26], [3]) for $p \leq 4$, $c > 1$, we have $D(\sigma, 0) \neq 0$ when $\operatorname{Re}\sigma > 0$ and hence the assumption (2.12) is met. Consequently, (2.14) follows from (4.2)

To handle the localized case, we note that when k, tends to zero, there is a single root $\lambda = 0$ of (8.7) on the imaginary axis and hence, there is spectrum of $A_\infty(\sigma, 0^+)$ on the imaginary axis. More precisely, for $k \sim 0$ this root behaves as

$$\mu(\sigma, k) \sim -\frac{k^2}{\sigma}. \tag{8.33}$$

Consequently, there is only one of the negative real part roots of (8.32) which goes to zero. Since $\mu(\sigma, k)$ is analytic, we can use the Gap lemma [11], [17] to get the continuation of the Evans function. Moreover, by using the same method as in the study of the gKP

equation, we can also write the Evans function as $|\tilde{D}(\sigma,0)| = |c\sigma D(\sigma,0)|$ where $D(\sigma,0)$ is the Evans function associated to the linearized BBM equation about the solitary wave. Again, since the BBM solitary wave is stable we also have that $\tilde{D}(\sigma,0)$ does not vanish for $\mathrm{Re}\,\sigma > 0$ and hence, (2.15) is verified. Note that, (4.8) is also met in view of (8.33) since

$$R(\sigma,k)J(ik)S(ik) = \partial_{xx}(-k^2\partial_x^{-2}) = -k^2.$$

Therefore, (2.16) follows from Lemma 4.3.

Finally, as for the gKP equation, the existence of an unstable eigenmode follows from Lemma 3.1. Indeed, we can write M_k under the form

$$M_k = (1-\partial_x^2)^{-1} m_k (1-\partial_x^2)^{-1},$$

where

$$m_k u = c\partial_x\Big(\partial_x(-\partial_{xx} + \frac{(c-1)}{c}\mathrm{Id} + \frac{p}{c}Q^{p-1}\mathrm{Id}\Big)\partial_x - k^2.$$

Again, the existence of a nontrivial kernel for m_k comes from the study of the KP equation and one can deduce that M_k is Fredholm from the fact that m_k is Fredholm.

Therefore, we can state the following result.

Theorem 8.5. *Consider the equation* (8.31) *for $c > 1$ and $p \leq 4$. For every $s \geq 0$, there exists $\eta > 0$ such that for every $\delta > 0$ there exists u_0^δ and a time $T^\delta \sim |\log\delta|$ such that $\|u_0^\delta - Q\|_{H^s(\mathbb{R}^2)} < \delta$ and the solution $u^\delta(t)$ of* (8.31) *with data u_0^δ is defined on $[0, T^\delta]$ with $u^\delta(t) - Q \in H^s(\mathbb{R}^2)$, $\forall t \in [0, T^\delta]$ and moreover satisfies the estimate*

$$\inf_{v\in\mathcal{F}} \|u^\delta(T^\delta) - v\|_{L^2(\mathbb{R}^2)} \geq \eta,$$

where $\mathcal{F}$ is the space of $L^2(\mathbb{R})$ functions independent of y.

A similar statement may be done for periodic in y solutions with a suitable period depending on the transverse frequency of the unstable mode (see Theorem 1 above).

Let us point out that the KP-BBM model considered in this section is not the relevant one from modelling view point (see [22]), the relevant one being

$$u_t - (c-1)u_x - u_{txx} + cu_{xxx} + \partial_x(u^p) + \partial_x^{-1}u_{yy} = 0. \tag{8.34}$$

Equation (8.34) does not fit in the framework considered in this paper and it is possible that the KdV soliton is in fact stable as a solution of (8.34). Nevertheless our KP-BBM model seems interesting for the following reason.

8.6. **Final remark.** Let observe that in the case $p = 2$ the equation (8.31) is globally well-posed for data close to Q. We have therefore nonlinear instability in the context of global well-posedness. Therefore this type of phenomena already encountered in the context of

the KP-I equation is not only restricted to integrable models as the KP-I equation. Let us briefly explain how we prove the global well-posedness for

$$u_t-(c-1)u_x-u_{txx}+cu_{xxx}+\partial_x(u^2)-\partial_x^{-1}u_{yy}=0$$

with initial data

$$u(0,x,y)=Q(x)+v_0(x,y),$$

where v_0 is localized both in x,y. More precisely, we suppose that $v_0\in H^s(\mathbb{R}^2)$ with s large enough. If we set $u=Q+v$ then we have that v solves the problem

$$v_t-(c-1)v_x-v_{txx}+cv_{xxx}+\partial_x(v^2)+\partial_x(Qv)-\partial_x^{-1}v_{yy}=0,\quad v(0,x,y)=v_0\,. \tag{8.35}$$

In the case $Q=0$ the above equation is shown to be globally well-posed in [29]. In the case of a Q which is bounded together with its derivatives one needs to combine the argument of [29] with the following control on the flow of (8.35). Multiplying (8.35) by v and integrating over $\mathbb{R}^2$ yields

$$\frac{d}{dt}\Big(\|v(t,\cdot)\|_{L^2}^2+\|\partial_x v(t,\cdot)\|_{L^2}^2\Big)=-2\int\partial_x(Qv)v=-\int Q'v^2\,.$$

A use of the Gronwall lemma provides the control

$$\|v(t,\cdot)\|_{L^2}+\|\partial_x v(t,\cdot)\|_{L^2}\leq(\|v_0\|_{L^2}+\|\partial_x v_0\|_{L^2})e^{c(1+|t|)}\,. \tag{8.36}$$

The local analysis of [29] shows that in the case $Q=0$ the problem (8.35) is locally well-posed for data such that $\|v_0\|_{L^2}+\|\partial_x v_0\|_{L^2}<\infty$. In order to include the term $\partial_x(Qv)$ in the local analysis of [29] one needs to evaluate the quantity

$$\|(1-\partial_x^2)^{-1}\partial_x(Qv)\|_{L_T^{1+\varepsilon}L^2_{x,y}}+\|(1-\partial_x^2)^{-1}\partial_x^2(Qv)\|_{L_T^{1+\varepsilon}L^2_{x,y}} \tag{8.37}$$

for some $\varepsilon>0$. The unessential loss ε (compared to the natural L^1_T coming from the Duhamel formula) is related to the fact that the well-posedness in [29] is established in Bourgain spaces and the non-linearity in a Bourgain's norm of type $X_T^{s,b-1}$, $b>1/2$ close to $1/2$ can be estimated by the non-linearity if $L_T^{1+\varepsilon}H^s$ with $\varepsilon>0$ close to zero. But the quantity (8.37) can be easily estimated in terms $\|v\|_{L_T^\infty L^2}+\|v_x\|_{L_T^\infty L^2}$ (and even only $\|v\|_{L_T^\infty L^2}$) which shows that the term $\partial_x(Qv)$ can be incorporated in the local analysis of [29] which in turn thanks to the control (8.36) implies that in the case $p=2$ the equation (8.31) is globally well-posed for data which is a localized perturbation of Q.

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